Identification of Synergies by Optimization of Trajectory Tracking Tasks

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Abstract— **According to the model of muscle synergies, the central nervous system (CNS) is organised in a modular structure, such that any muscle activation can be produced as a linear superposition of predefined time-varying profiles (i.e. synergies). This organisation might contribute to simplify the control of the musculoskeletal apparatus. Taking inspiration from these findings, we propose a method to identify the synergies that can be used to control a given dynamical system for the task of tracking a set of trajectories. Further, we show how the same approach can be applied to assess the impact of the number of synergies on the performance of the control method. From the theoretical point of view, we provide a novel interpretation of synergies inspired by the Karhunen-Loève decomposition; furthermore, our method suggests that the quality of a set of synergies should be measured in task space rather then in input space.**

I. INTRODUCTION

One of the current trends in human motor control is the idea that the CNS generates muscle activations by combining a finite number of muscle synergies, i.e. activations of muscles with specific time-varying profiles [1]. There is evidence that weighted linear combinations of these elementary controls can account for a variety of tasks [4], [10].

From the computational point of view, the concept of muscle synergies is very attractive [3]. Admissible controls are restricted to the vector space spanned by a finite number of elementary inputs. As a result, the control of an inherently non-linear system like the human musculoskeletal apparatus is tackled by employing a linear approach. Furthermore, learning a task reduces to finding the right combination of muscle synergies (i.e. weights for their linear combination) instead of finding a time sequence of input to each muscle.

This work is a first step towards a control scheme that takes inspiration from the idea of muscle synergies. We propose a method to identify the synergies that can optimally drive a given mechanical system along a set trajectories. From the theoretical point of view, this work contributes a novel interpretation of muscle synergies that is inspired by the Karhunen-Loève decomposition [14], and that capitalizes in assessing the quality of these control actions in task space (in terms of tracking error) rather then in inputs space (as it is typically done in neuroscience [1], [4]).

This paper is organised as follows. Sec. II describes some related work. Sec. III presents the mathematical details of the method proposed for the identification of muscle synergies.

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Sec. IV shows how the method can be used to study the performance of the controller as a function of the number of synergies. Sec. V details the experiments performed and Sec. VI shows the results. Finally Sec. VII derives the conclusions, presents a short discussion and states future research directions.

II. RELATED WORK

In neuroscience, muscle synergies are typically identified by applying a decomposition method (e.g. PCA , ICA, non-negative matrix factorization) to a dataset of EMG data extracted from a group of subjects while they are performing a given set of actions. The goal is to find the generators of the vector space that best approximate the dataset of EMG signals. The number of synergies to be extracted is not known in advance and it is typically an input parameter to the decomposition algorithm.

In robotics and control engineering, some effort has been made to develop control strategies inspired by the concept of muscle synergies. Besides the technological aspects, these studies also provide computational approaches to interpret, model and test the idea of muscle synergies. In [7] the authors derive analytically an optimal set of primitives for a system previously feedback-linearised; this method applies if the system is feedback linearisable and its dynamical model is known in analytical form. Other researchers approximate the dynamics of the system with a lower order linear dynamical model, and obtain a set of primitives by algebraic considerations on its matrices [11]. Another approach consists of applying a learning procedure to a training set of sensory-motor data generated by actuating the robot with random pulses [8], [12]. Although it bypasses the limitations of the analytical formulation, this method does not provide a formal definition of the extracted synergies.

III. IDENTIFICATION OF SYNERGIES

The interpretation of muscle synergies presented in this paper, as well as the method we propose for their identification, generalises the Karhunen-Loève decomposition [14] to the input-output relation of a dynamical system. To make this clear, in this section we first give a general definition of synergies (Sec. III-A), and then, after a brief discussion on this decomposition (Sec. III-B), we detail the method proposed to compute optimal synergies for control (Sec. III-C).

A. Definition of synergies

In their most general connotation, synergies constitute a finite set of control actions (i.e. input to the system to be

controlled) that, by linear combinations, lead to a specific set of output trajectories.

Let us consider a dynamical system

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}) + g(\boldsymbol{x})\boldsymbol{u} \qquad (1)$$
$$\boldsymbol{y} = h(\boldsymbol{x})$$

where $\boldsymbol{u} \in \mathbb{R}^m$ is the input vector and $\boldsymbol{y}$ is the output.

In accordance with the definition [1], we model each synergy as a function of time, $\boldsymbol{\phi}_i(t)$. Any valid input sequence $\boldsymbol{u}(t)$ is assumed to be generated as a linear combination of these functions[1]

$$\boldsymbol{u}(t) = \sum_{i=1}^{k} c_i \boldsymbol{\phi}_i(t) \qquad \boldsymbol{\phi}_i : \mathbb{R}^+ \to \mathbb{R}^m \qquad (2)$$

where $c_i \in \mathbb{R}$. Given a set of k synergies $\{\boldsymbol{\phi}_i(t)\}_{i=1...k}$, defined to be task-independent, the weights $\{c_i\}_{i=1...k}$ encode a specific input sequence and therefore, depending on the initial conditions of the system, the corresponding output trajectory.

The synergies $\{\boldsymbol{\phi}_i(t)\}_{i=1...k}$ are the generators of the vector space of the admissible control actions. Therefore, the number and the shape of these elementary inputs have to be chosen appropriately to maximise the range of control that can be obtained.

B. Karhunen-Loève decomposition

Let's consider a stochastic process $\{X_t\}_{t \in [a,b]}$ with null expected value (i.e. $E[X_t] = 0, \forall t \in [a,b]$). It can be proven that X_t can be represented by means of another stochastic process $\hat{X}_t$ that is defined as the infinite linear combination of deterministic orthonormal functions $\{\phi_1(t), \phi_2(t), \dots\}$ in $L^2([a,b])$:

$$X_t \sim \hat{X}_t \qquad \text{with} \qquad \hat{X}_t = \sum_{i=1}^{\infty} c_i \phi_i(t) \qquad (3)$$

where the stochastic coefficients are chosen so as to minimise a suitable distance between X_t and $\hat{X}_t$:

$$\mathbf{c}(\phi_1, \phi_2 \dots) = \underset{c_1, c_2, \dots}{\operatorname{argmin}} \; d(X_t, \hat{X}_t) \qquad (4)$$

Truncating the linear combination to k components, $\hat{X}_t$ becomes an approximation of X_t; the quality of this approximation depends on the employed ϕ_i:

$$d^*(\phi_1, \dots, \phi_k) = \underset{c_1, \dots, c_k}{\min} \; d(X_t, \hat{X}_t). \qquad (5)$$

Thus, it makes sense to compute the functions ϕ_i that results in the minimum error:

$$[\phi_1^*, \dots, \phi_k^*] = \underset{\phi_1, \dots, \phi_k}{\operatorname{argmin}} \; d^*(\phi_1, \dots, \phi_k). \qquad (6)$$

It can be shown that the best basis $\{\phi_i(t)\}_{i=1...k}$ (in the sense of the mean-squared error between X_t and $\hat{X}_t$) is the Karhunen-Loève decomposition, which correspond to the

well know principal component analysis (PCA) when the stochastic process $\{X_t\}_{t \in [a,b]}$ is replaced by the discrete and finite time process $\{X_n\}_{n=1...N}$ [14].

C. Interpretation, identification and testing of synergies

Let us specify the shape of each synergy by means of a set of parameters[2], $\boldsymbol{\phi}_i(t, a_0^i, ..., a_r^i)$. Further, let us call $\hat{\boldsymbol{y}}(\boldsymbol{x}_0, \boldsymbol{u}(t))$ the output trajectory of the system, obtained from the initial condition $\boldsymbol{x}_0$ and the input sequence $\boldsymbol{u}(t)$.

If $\{\boldsymbol{y}_1(t), ..., \boldsymbol{y}_n(t)\}$ is a set desired output trajectories (used as training set), identifying the appropriate synergies translates into finding the parameters $\{a_0^i, ..., a_r^i\}_{i=1..k}$ such that linear combinations of the corresponding $\boldsymbol{\phi}_i(t)$ lead to the best approximation of the desired set. Thus, we can define the quality measure of the set of synergies as follows:

$$d(\boldsymbol{y}, \hat{\boldsymbol{y}}) = \sum_{j=1}^{n} \left\| \hat{\boldsymbol{y}} \left(\boldsymbol{x}_0^j, \sum_i c_i^j \boldsymbol{\phi}_i(t) \right) - \boldsymbol{y}_j(t) \right\| \qquad (7)$$

where $\|\cdot\|$ is an appropriately chosen norm of a function. The expression within norm represents the error between the output of the system and the j-th desired trajectories (i.e. tracking error). It is worth noting that while the weights c_i depend on the trajectories to be approximated (i.e. they are task-dependent), the set of synergies accounts for the entire desired set. Further, we assume that $\boldsymbol{y}_i(0) = h(\boldsymbol{x}_0^i)$.

The identification of the parameters (learning phase) can be performed by executing the following minimization:

$$[\boldsymbol{\phi}_1, \dots, \boldsymbol{\phi}_k, \boldsymbol{c}] = \underset{\substack{a_0^1, \dots, a_r^k \\ c_1^1, \dots, c_k^n}}{\operatorname{argmin}} \; d(\boldsymbol{y}, \hat{\boldsymbol{y}}) \qquad (8)$$

where, in the left hand side, we enclosed all the synergy-combinators in the vector $\boldsymbol{c}$, and we substituted the synergy-coefficients with the corresponding $\boldsymbol{\phi}_i$.

It now becomes easy to notice the similarity between Eq. 5-6 and Eq. 8. More precisely, X_t is equivalent to the set of desired trajectories, and $\hat{X}_t$ is equivalent to the output that can be obtained from linear superposition of elementary controls. In other words, as the Karhunen-Loève functions are the components that better approximate the random process X_t, we interpret muscle synergies as the elementary controls that better drive the system (1) along the set of trajectories $\boldsymbol{y}_i$ (in the sense of the norm between $\boldsymbol{y}$ and $\hat{\boldsymbol{y}}$).

The synergies identified in the training phase should then be tested for generalization (testing phase). The idea is to evaluate to which extent they can generate inputs that drive the system along a new group of trajectories (testing set). An optimization similar to Eq. (8) is used to identify the synergy-combinators c_i^j that minimise the error in tracking each of the testing trajectories; the parameters a are kept constant to the values identified in the training phase. The obtained errors are used as a measure of the generality of the set of synergies.

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[1]As presented in [1], $\boldsymbol{u}(t)$ is obtained by linear combination and time-shift of synergies. In this preliminary work we only consider linear combination in order to simplify the mathematical formulation.

[2]For clarity reason in the rest of the manuscript the dependency on the parameters is omitted.

The method to perform the optimization as well as the norm used to compute the errors can be chosen by the designer.

IV. MINIMUM NUMBER OF SYNERGIES

The number of synergies defines the dimensionality of the new input space, therefore a small number of these elementary controls is a desired feature of the controller. The optimization method (8) can also be used to investigate the minimum number of synergies required to obtain a certain tracking accuracy. By performing the same minimization (i.e. on the same dynamical system and the same desired trajectories) for different number of synergies, it is indeed possible to derive the trend of the error as a function of the number of these elementary controls. In fact, while for some linear systems this question can be addressed by mathematical proof, it could be problematic to do so for non-linear systems.

One of the challenges in employing any local optimization-based method is the issue of the local minima; the minimization algorithm could discover a local minima of the objective function instead of a global one. From a practical point of view, sub-optimal solutions might still lead to acceptable tracking performance (see Sec. VI). However, if the goal is to compare the performance obtained with different number of synergies, local solutions are undesirable. Sure enough, if the algorithm outputs sub-optimal solutions, the trend of the error as a function of the number of synergies could be an artefact. For example, assuming that the error should not become larger increasing the number of synergies, the local minimum obtained with n+1 synergies might be higher than the one obtained with n synergies, while the global minima follow the expected trend.

Elaborating on these consideration we have developed an algorithm that, although does not solve the problem of the local minima (i.e. the solution computed can be sub-optimal), seems to compute the qualitatively correct trend of the error (see Sec. VI-A). The main intuition is that by starting the minimization with n+1 synergies from the solution obtained in the previous step (i.e. with n synergies) the error should not increase. Indeed, since the additional parameters add approximation power to the algorithm and the previous solution already lies on a minimum, either the gradient of the objective function stays zero or the previous solution can move towards a smaller objective function value (leading to a decrement of the error). To avoid that, due to a flat gradient (e.g. saddle point), the values of the parameters to be optimised do not change, 15 optimizations are performed in parallel starting from different initial conditions around the previous solution. The initial conditions are extracted from a multivariate Gaussian random distribution centred in the previous solution and having a standard deviation equal to 0.005 on each dimension (diagonal covariance matrix). The parameters leading to the lowest objective function value are considered the solution of this iteration and are used

as initial conditions for the next one. Algorithm 1 describes this procedure in pseudo-code.

Algorithm 1 ERRORTREND(f, k, ns, σ)

Require: The objective function f
 The maximum number of synergies k
 The number of starting points ns
 The standard deviation σ.
Ensure: The tracking errors err.

1: $initConds \leftarrow$ RAND(ns)
2: $allSolutions \leftarrow$ MINIMISEALL($f, initConds$)
3: $err[1] \leftarrow$ MIN($allSolutions$)

4: **for** $i = 2 \rightarrow k$ **do**
5:    $initConds \leftarrow$ RANDGAUSS($err[i-1], \sigma, ns$)
6:    $allSolutions \leftarrow$ MINIMISEALL($f, initConds$)
7:    $err[i] \leftarrow$ MIN($allSolutions$)
8:    $i \leftarrow i + 1$
9: **end for**

V. EXPERIMENTAL METHODS

A. Dynamical systems

The method proposed in Sec. III-C is evaluated in simulation for both a linear and a non-linear dynamical system. Although the systems are quite simple, they capture some interesting features that are worth studying since they are present both in the human musculoskeletal apparatus and in different types of robots; i.e. redundancy, compliance and trigonometric non-linearities.

a) Linear system: The linear system resembles an agonist-antagonist pair of compliant actuators applied on a mass m; since the forces produced by the linear springs always lie on the same direction, the mass can only move in one dimension. This system (see Fig. 1) can be described by the following mathematical model:

$$\ddot{x} = \frac{1}{m} \left(-k_1(x - x_1) - k_2(x_2 - x) - c\dot{x} \right) \quad (9)$$

where k_1 and k_2 are the spring constants and c is the coefficient of a dissipative force. The output of the system is chosen to be the position x of the mass, while the position of the spring extremities represent the inputs x_1 and x_2. The system is linear, compliant and redundant (i.e. the number of inputs exceeds the number of the output variables).
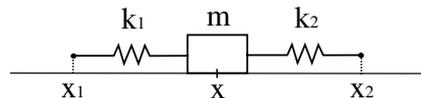

Fig. 1: Agonist-antagonist linear system.

b) Non-linear system: The non-linear system is the well-known pendulum, that can be trivially described by the following dynamical model

$$\ddot{\vartheta} = -\frac{g}{r} \sin(\vartheta) + \frac{\tau}{r} \quad (10)$$

where g is the gravitational acceleration and r the length of the rod. The output of the system, ϑ, is the angle between the rod and the vertical axis, and the input τ is the torque applied on the revolute joint. This system has been chosen because it is characterised by the same non-linearities as each kinematic chain, such as most robotic manipulators (i.e. trigonometric non-linearities). Moreover, because of its simplicity, it is a good starting point to investigate the effectiveness of controlling a non-linear plant by linear combinations of elementary inputs.

B. Desired trajectories

The tasks that the controlled systems have to accomplish consist in tracking a set of desired trajectories. Each of these is defined as the smoothest trajectory to reach a final position from an initial position in a given amount of time. As described in [6], smoothness can be quantified by the third derivative of the trajectory itself (i.e. jerk). There is evidence that minimum-jerk trajectories can be used to model the hand-motion of a human subject who is instructed to bring his hand from an starting to an ending position in a certain amount of time [5]. Assuming the movement to start and end with zero velocity and acceleration (at time t_0 and t_f respectively), the following expression describes the minimum-jerk trajectory of the coordinate x from x_0 to x_f:

$$x(t) = x_0 + (x_0 - x_f)(15\rho^4 - 6\rho^5 - 10\rho^3) \qquad (11)$$

were $\rho = \frac{t}{t_f - t_0}$.

For each experiment, both training and testing set consist of minimum-jerk trajectories characterised by 1 s of duration ($t_0 = 0$ and $t_f = 1$) and 100 time samples. Initial and final positions are specified in Sec. VI. Mathematically, the set of desired outputs is restricted to the subspace of L^2 of the minimum-jerk trajectories.

C. Synergy model

Both for the linear and the non-linear systems, each synergy consists of as many parametrised functions as the number of input variables (i.e. two for the agonist-antagonist pair and one for the pendulum). For the purposes of this paper these functions are defined as 5-th order polynomials:

$$\phi(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 \qquad (12)$$

therefore, if m is the number of input variables, for each synergy $6m$ parameters need to be identified. The reason of this choice is that for the linear dynamical system (9), 5-th order polynomials can span the whole output space of minimum-jerk trajectories (see the appendix for the mathematical proof). This choice does not affect the generality of the method described in Sec. III-C.

The optimization described in Sec. III-C, Eq. (8), is performed numerically in Matlab® (R2011a, Mathworks, Inc.) using the fmincon solver and the interior-point algorithm [9].

In Eq. (7), the error between the desired trajectories and the output of the system is computed using Euclidean norm.

VI. RESULTS

A. Agonist-antagonist linear pair

The results presented in this sections are obtained with the following parameters of the dynamical model: $m = 1$ Kg, $k_1 = k_2 = 1$ N/m, and $c = 1.5$ Ns/m. The three training minimum-jerk trajectories are defined by the initial positions 60, 120 and 180 m, and the final position 50 m. Forty-two testing trajectories are defined as follows. The interval $[30; 180]$ m is discretised in 7 equally (and maximally) spaced points; any pair of different points corresponds to the initial and final position of a testing minimum-jerk trajectory.

Fig. 2a depicts the trend of the error in tracking the desired training trajectories; the input to the system consists of appropriate linear combinations of the extracted synergies. This plot is computed using Algorithm 1. It can be noted that there is a drastic decrement of the error switching from one to two synergies; the use of any additional synergy does not seem to play a role in minimising the tracking performance. This result is confirmed by mathematical proof (see appendix).

While Fig. 2a describes qualitatively the expected theoretical results (i.e. two synergies are enough to obtain the best possible tracking performance), quantitatively the minimisation generates a sub-optimal solution; the lowest tracking error is equal to 10^{-7} while an exact mathematical solution should be characterised by an error equal to 0 (i.e. 10^{-16} considering the precision of the machine used and the numerical minimization error).

Practically, the obtained sub-optimal solution leads to satisfactory tracking performance (error = 10^{-3} m). Table I reports the optimal coefficients of the two synergies as obtained by optimisation (8). The similarity between the coefficients of ϕ_{i1} and ϕ_{i2} is due to the inherent symmetry of the mechanical system (see Fig. 1). Fig. 2b depicts the three training trajectories (continuous lines) and the corresponding outputs of the system (dashed lines).

	a_0	a_1	a_2	a_3	a_4	a_5
ϕ_{11}	15.48	21.36	75.02	-195.10	88.47	-5.95
ϕ_{12}	19.08	21.82	68.95	-193.58	84.27	-1.76
ϕ_{21}	0.49	0.62	2.04	-5.46	2.43	-6.17
ϕ_{22}	0.48	0.60	2.00	-5.47	2.43	-6.15

TABLE I: Optimal synergies identified for the agonist-antagonist linear system. The coefficient ϕ_{ij} indicates the j-th element of the synergy i.

The results obtained in the test of generalization confirm that two synergies span the entire output space of minimum-jerk trajectories (see appendix). The tracking errors for the 42 testing trajectories are distributed in the order of magnitude of 10^{-4} m (see Fig. 3a). Fig. 5b shows the tracking performance for the best tracked (x_0=180, x_f=155 m) and the worst tracked ($x_0 = 30$, $x_f = 180$ m) testing trajectories.

B. Pendulum

To obtain the results presented in this section, the parameters of the dynamical model are: $r = 0.1$ m and $g =$
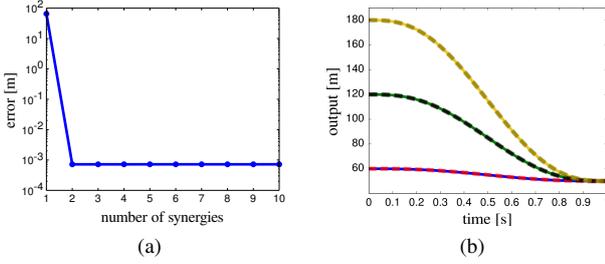
Fig. 2: Trend of the error as a function of the number of synergies (a), and performance of tracking the training trajectories (continuous lines) for the agonist-antagonist linear system.
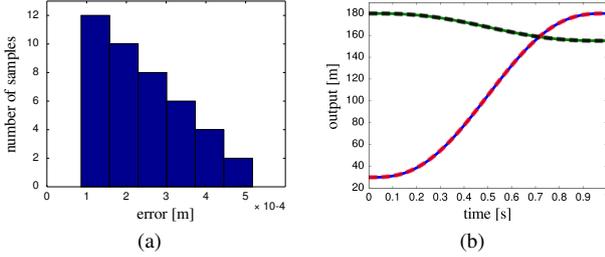

Fig. 3: Distribution of the tracking errors for the testing trajectories (a). Real (dashed lines) and desired (continuous lines) output for the best and the worst-tracked testing trajectories (b) in the agonist-antagonist linear system.

	a_0	a_1	a_2	a_3	a_4	a_5
ϕ_1	4.83	4.45	2.09	4.86	2.61	-1.57
ϕ_2	-0.38	-0.32	0.35	-0.03	-0.07	-0.39
ϕ_3	-0.30	0.70	-0.28	-0.23	-0.02	0
ϕ_4	-0.06	-0.03	-0.22	-0.18	-0.05	-0.06
ϕ_5	-0.10	-0.16	0.27	0	-0.06	0.02

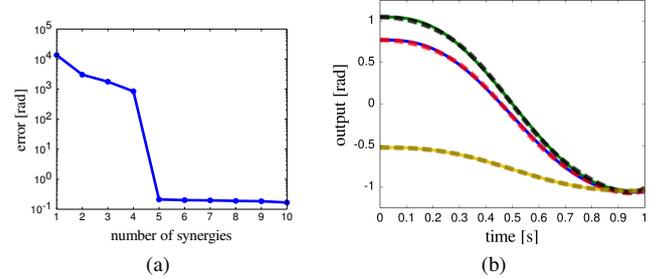TABLE II: Optimal synergy coefficients identified for the non-linear system (pendulum).

Fig. 4: Trend of the error as a function of the number of synergies (a), and performance of tracking the training trajectories (continuous lines) for the non linear system (pendulum).

9.81 m/s². The three training trajectories are characterised by the initial positions $\pi/4$, $\pi/3$ and $-\pi/6$ rad, and the final position $-\pi/3$ rad. Forty-two testing trajectories are defined as follows. The interval $\left[0; \frac{\pi}{2}\right]$ rad is discretised in 7 equally (and maximally) spaced points; any pair of different points corresponds to the initial and final positions of a testing minimum-jerk trajectory. The interval $\left[\frac{\pi}{2}, \pi\right]$ is not considered because it would lead to the same coefficients c_i (apart from a minus sign), and therefore practically to the same control actions.

Fig. 4a shows the evolution of the tracking error as a function of the number of synergies. After having reached the value of 10^{-1} rad (i.e. 10^{-3} rad in average for each sample) with 5 synergies, the introduction of new elementary controls does not significantly affects the error.

Table II summarises the coefficients of the 5 synergies identified by the algorithm on the training-set. The performance of the system in tracking the training-set can be seen in Fig. 4b. The output of the system (dashed lines) is generally close to the corresponding desired behaviour (continuous lines) for all the trajectories.

Unlike the linear case (see Sec. VI-A), for this non-linear system there is no mathematical evidence of the minimum number of synergies required to span the whole output space of minimum-jerk trajectories. However, the results of the generalization tests provide evidence that 5 synergies can lead to good performance in tracking minimum-jerk trajecto-

ries that do not belong to the training set. The distribution of the errors shows that the most of the testing trajectories can be tracked with an error of 10^{-2} rad in order of magnitude (see Fig. 5a). Fig. 5b shows the tracking performance for the best tracked (x_0=0.52, x_f=1.04 rad) and the worst tracked ($x_0 = 1.57$, $x_f = 0$ rad) testing trajectories.
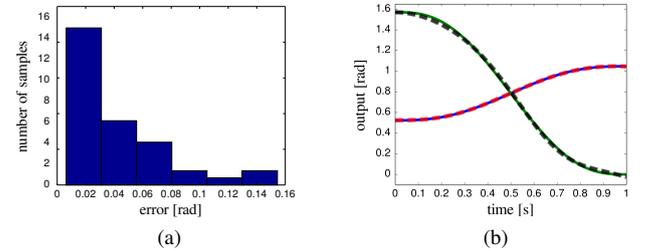

Fig. 5: Distribution of the tracking errors for the testing trajectories (a). Real (dashed lines) and desired (continuous lines) output for the best and the worst-tracked testing trajectories (b) in the non linear system (pendulum).

VII. CONCLUSIONS AND FUTURE WORK

A. Conclusions

This paper proposes a new interpretation of muscle synergies inspired by the Karhunen-Loève decomposition, and a method to identify a set of synergies that can be used to control a given dynamical system. Any admissible input to the system is expressed as a linear combination of these elementary controls, defined as parametrised functions of time. The method proposed consists of optimising the set

of parameters that define each synergy with the goal of minimising the error between the output of the system, controlled by appropriate linear superposition of the synergies, and a set of desired training trajectories.

The method to identify the synergies has been evaluated in simulation for a linear and a non-linear system. The performance in tacking the desired trajectories are satisfactory. Moreover, the results obtained in the test of generalization show that the synergies identified lead to good tracking performance on minimum-jerk trajectories not used for the training. The algorithm proposed to derive the trend of the tracking error as a function of the number of synergies seems to produce results compatible with the theory and it has been used to establish the number of synergies to be employed for each of the experiments.

B. Discussion

The idea underlying the model of muscle synergies is to approximate the (non-linear) vector space of muscle activations (i.e. input space) with a linear vector space defined by the synergies (that serve as generators). The quality of the identified synergies is typically measured in the input space as the error in approximating a dataset of EMG data [1], [10], [4]. In this paper we took a different approach by proposing a (PCA-like) interpretation of synergies that minimizes approximation-errors directly in the task space (e.g. errors in tracking a given set of trajectories). This difference is quite relevant as there exist mechanical systems (e.g. unstable systems) that produce very different output-trajectories in response to very small differences in the input space. Furthermore, there is evidence that, in humans, control actions are formulated in task space [13].

The similarity between our method and the Karhunen-Loève decomposition can be summarised as follows. As the latter finds the functions that best approximate a stochastic process, the method we propose seeks the synergies that produce the best approximation of a set of desired output trajectories. Clearly, the scopes of the two methods are different: the former operates on stochastic processes, the latter operates on the input-output relationship of dynamical systems. Although in this paper we have only considered deterministic systems, the human musculoskeletal apparatus is inherently noisy [15]. Form this point of view, our interpretation of muscle synergies becomes even more related to the Karhunen-Loève decomposition, as the output trajectories of a noisy dynamical system can be considered stochastic processes.

The main advantage of employing a synergy-like controller is the dimensionality reduction it follows. According to the *time-varying* synergy model [1] employed in this work, an input sequence that potentially belongs to an infinite dimensional space (e.g. the space of continuous vector valued functions), is obtained by superimposing linearly a finite number of time-varying synergies. As opposed to a time-variant sequence of input to each actuator, the new input to the system is the finite set of time-invariant synergy-combinators. Noteworthy, even if there might be a larger number of synergies then actuators (as in the case of the pendulum – see Sec. VI), the dimensionality is anyway reduced due to the shift from time-variant to time-invariant input signals. The same would not hold if we employed the so called *synchronous-synergy* model [2]; i.e. muscle synergies are constant coefficients and the synergy-combinators are time-variant input signals. In that model, to obtain a dimensionality reduction the number of synergies needs to be lower then the number of actuators. Additionally, it is worth reminding that the same set of synergies generates a wide repertoire of motions (i.e. minimum-jerk trajectories). Especially in the case of non-linear dynamics, this consideration justifies the number of required synergies, that might appear high if compared to the dimensionality of the state space (see Sec. VI-B). The identification procedure basically requires to solve the control problem for the training trajectories; however, once the appropriate synergies are identified, learning a new motion reduces to finding only the synergy-combinators, simplifying substantially the subsequent learning problems.

Similarly to the work presented in Sec. II, this paper proposes a method to synthesise synergies for control. In line with the analytical procedures [7], [11], our method gives a clear mathematical interpretation of muscle synergies; however, it does not require the dynamical model of the system in analytical form. Moreover, unlike [11], [12], our synergies are optimised to perform trajectory tracking, rather then reaching tasks.

C. Future Work

Given the success of this preliminary study, we believe that improvements to the numerical method to identify muscle synergies will be the core of our future work. This will allow us to evaluate our interpretation of synergies in more complex biologically-inspired systems (e.g. kinematic chains with muscle-like actuators).

The method proposed here will be extended to track generic trajectories in L^2. Most probably this will require to model synergies with universal function approximators; indeed, deriving in advance the appropriate analytical form of the synergies might not be trivial for non-linear dynamical systems and generic desired trajectories.

Finally, we will make an effort to introduce feedback in the control paradigm and to examine its impact in the set of synergies previously identified [2].

APPENDIX

Proposition 1: For the dynamical system (9), two input synergies consisting of 5-th order polynomials span the whole output space of minimum-jerk trajectories (11).

Proof: Let us compute the first and the second derivative of (11)

$$\dot{x} = (x_0 - x_f)(60\rho^3 - 30\rho^4 - 30\rho^2) \qquad (13)$$

$$\ddot{x} = (x_0 - x_f)(180\rho^2 - 120\rho^3 - 60\rho) \qquad (14)$$

and define a new variable $s = x_0 - x_f$. Substituting (11), (13) and (14) in (9), and expressing the input variables x_1

and x_2 by polynomials of 5-th order, we obtain[3]

$$s(180\rho^2 - 120\rho^3 - 60\rho) = \qquad (15)$$
$$- 2\left[x_0 + s(15\rho^4 - 6\rho^5 - 10\rho^3)\right] +$$
$$+ s(45\rho^4 + 45\rho^2 - 90\rho^3) +$$
$$+ a_0 + a_1t + a_2t^2 + a_3t^3 + a_4t^4 + a_5t^5 +$$
$$+ b_0 + a_1t + b_2t^2 + b_3t^3 + b_4t^4 + b_5t^5$$

To achieve an analytical solution, the coefficients of the unknown polynomials have to account for the corresponding terms of the equation. In particular, the polynomial obtained by the sum of the unknown polynomials has to be

$$-12s\rho^5 - 15s\rho^4 - 50s\rho^3 + 135s\rho^2 - 60s\rho + 2x_0 \quad (16)$$

Expression (16) can be rewritten as

$$\begin{bmatrix} s & x_0 \end{bmatrix} \begin{bmatrix} -12\rho^5 - 15\rho^4 - 50\rho^3 + 135\rho^2 - 60\rho \\ 2 \end{bmatrix}$$

where the vector $[s, x_0]$ is the only term that depends on the desired trajectory. Thus, the problem of finding a set of synergies that span the whole output space of the minimum-jerk trajectories translates into finding a basis of the space $\mathbb{R}^2$. Having dimension 2, all the basis of the space $\mathbb{R}^2$ consist of two linearly independent vectors. ∎

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[3]Without loss of generality here we considered $m = 1$, $k_1 = k_2 = 1$ and $c = 1.5$.