



Introduction to Complex Numbers

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The aim of this package is to provide a short study and self assessment programme for students who wish to become more familiar with complex numbers.

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The full range of these packages and some instructions, should they be required, can be obtained from our web page [Mathematics Support Materials](#).

1. The Square Root of Minus One!

If we want to calculate the square root of a negative number, it rapidly becomes clear that neither a positive or a negative number can do it.

$$\text{E.g., } \sqrt{-1} \neq \pm 1, \text{ since } 1^2 = (-1)^2 = +1.$$

To find $\sqrt{-1}$ we introduce a new quantity, i , defined to be such that $i^2 = -1$. (Note that engineers often use the notation j .)

Example 1

$$\begin{aligned} \text{(a)} \quad \sqrt{-25} &= 5i \\ \text{Since } (5i)^2 &= 5^2 \times i^2 \\ &= 25 \times (-1) \\ &= -25. \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \sqrt{-\frac{16}{9}} &= \frac{4}{3}i \\ \text{Since } \left(\frac{4}{3}i\right)^2 &= \frac{16}{9} \times (i^2) \\ &= -\frac{16}{9}. \end{aligned}$$

2. Real, Imaginary and Complex Numbers

Real numbers are the usual positive and negative numbers.

If we multiply a real number by i , we call the result an *imaginary* number. Examples of imaginary numbers are: i , $3i$ and $-i/2$.

If we add or subtract a real number and an imaginary number, the result is a *complex* number. We write a complex number as

$$z = a + ib$$

where a and b are real numbers.

3. Adding and Subtracting Complex Numbers

If we want to *add* or *subtract* two complex numbers, $z_1 = a + ib$ and $z_2 = c + id$, the rule is to add the real and imaginary parts separately:

$$z_1 + z_2 = a + ib + c + id = a + c + i(b + d)$$

$$z_1 - z_2 = a + ib - c - id = a - c + i(b - d)$$

Example 2

$$(a) \quad (1 + i) + (3 + i) = 1 + 3 + i(1 + 1) = 4 + 2i$$

$$(b) \quad (2 + 5i) - (1 - 4i) = 2 + 5i - 1 + 4i = 1 + 9i$$

EXERCISE 1. Add or subtract the following complex numbers. (Click on the green letters for the solutions.)

$$(a) \quad (3 + 2i) + (3 + i) \qquad (b) \quad (4 - 2i) - (3 - 2i)$$

$$(c) \quad (-1 + 3i) + \frac{1}{2}(2 + 2i) \qquad (d) \quad \frac{1}{3}(2 - 5i) - \frac{1}{6}(8 - 2i)$$

Quiz To which of the following does the expression

$$(4 - 3i) + (2 + 5i)$$

simplify?

(a) $6 - 8i$

(b) $6 + 2i$

(c) $1 + 7i$

(d) $9 - i$

Quiz To which of the following does the expression

$$(3 - i) - (2 - 6i)$$

simplify?

(a) $3 - 9i$

(b) $2 + 4i$

(c) $1 - 5i$

(d) $1 + 5i$

4. Multiplying Complex Numbers

We *multiply* two complex numbers just as we would multiply expressions of the form $(x + y)$ together (see the package on **Brackets**)

$$\begin{aligned}(a + ib)(c + id) &= ac + a(id) + (ib)c + (ib)(id) \\ &= ac + iad + ibc - bd \\ &= ac - bd + i(ad + bc)\end{aligned}$$

Example 3

$$\begin{aligned}(2 + 3i)(3 + 2i) &= 2 \times 3 + 2 \times 2i + 3i \times 3 + 3i \times 2i \\ &= 6 + 4i + 9i - 6 \\ &= 13i\end{aligned}$$

EXERCISE 2. Multiply the following complex numbers. (Click on the green letters for the solutions.)

(a) $(3 + 2i)(3 + i)$

(b) $(4 - 2i)(3 - 2i)$

(c) $(-1 + 3i)(2 + 2i)$

(d) $(2 - 5i)(8 - 3i)$

Quiz To which of the following does the expression

$$(2 - i)(3 + 4i)$$

simplify?

(a) $5 + 4i$

(b) $6 + 11i$

(c) $10 + 5i$

(d) $6 + i$

5. Complex Conjugation

For any complex number, $z = a + ib$, we *define* the complex conjugate to be: $z^* = a - ib$. It is very useful since the following are real:

$$z + z^* = a + ib + (a - ib) = 2a$$

$$zz^* = (a + ib)(a - ib) = a^2 + iab - iab - a^2 - (ib)^2 = a^2 + b^2$$

The *modulus* of a complex number is defined as: $|z| = \sqrt{zz^*}$

EXERCISE 3. Combine the following complex numbers and their **conjugates**. (Click on the **green** letters for the solutions.)

(a) If $z = (3 + 2i)$, find $z + z^*$ (b) If $z = (3 - 2i)$, find zz^*

(c) If $z = (-1 + 3i)$, find zz^* (d) If $z = (4 - 3i)$, find $|z|$

Quiz Which of the following is the modulus of $4 - 2i$?

(a) $\sqrt{20}$

(b) 2

(c) 20

(d) $\sqrt{12}$

6. Dividing Complex Numbers

The *trick* for dividing two complex numbers is to multiply top and bottom by the complex conjugate of the denominator:

$$\frac{z_1}{z_2} = \frac{z_1}{z_2} = \frac{z_1}{z_2} \times \frac{z_2^*}{z_2^*} = \frac{z_1 z_2^*}{z_2 z_2^*}$$

The denominator, $z_2 z_2^*$, is now a real number.

Example 4

$$\begin{aligned}\frac{1}{i} &= \frac{1}{i} \times \frac{-i}{-i} \\ &= \frac{-i}{i \times (-i)} \\ &= \frac{-i}{1} \\ &= -i\end{aligned}$$

Example 5

$$\begin{aligned}\frac{(2+3i)}{(1+2i)} &= \frac{(2+3i)(1-2i)}{(1+2i)(1-2i)} \\ &= \frac{(2+3i)(1-2i)}{1+4} \\ &= \frac{1}{5}(2+3i)(1-2i) \\ &= \frac{1}{5}(2-4i+3i+6) = \frac{1}{5}(8-i)\end{aligned}$$

EXERCISE 4. Perform the following **divisions**: (Click on the **green** letters for the solutions.)

(a) $\frac{(2+4i)}{i}$

(b) $\frac{(-2+6i)}{(1+2i)}$

(c) $\frac{(1+3i)}{(2+i)}$

(d) $\frac{(3+2i)}{(3+i)}$

Quiz To which of the following does the expression

$$\frac{8 - i}{2 + i}$$

simplify?

(a) $3 - 2i$

(b) $2 + 3i$

(c) $4 - \frac{1}{2}i$

(d) 4

Quiz To which of the following does the expression

$$\frac{-2 + i}{2 + i}$$

simplify?

(a) -1

(b) $\frac{1}{5}(-5 + 7i)$

(c) $-1 + \frac{1}{2}i$

(d) $\frac{1}{5}(-3 + 4i)$

7. Quiz on Complex Numbers

Begin Quiz In each of the following, simplify the expression and choose the solution from the options given.

1. $(3 + 4i) - (2 - 3i)$
(a) $3 - i$ (b) $5 + 7i$
(c) $1 + 7i$ (d) $1 - i$
2. $(3 + 3i)(2 - 3i)$
(a) $6 - 8i$ (b) $6 + 8i$
(c) $-3 + 3i$ (d) $15 - 3i$
3. $|12 - 5i|$
(a) 13 (b) $\sqrt{7}$
(c) $\sqrt{119}$ (d) -12.5
4. $(7 - 17i)/(5 - i)$
(a) $\frac{7}{5} + 17i$ (b) $3 + i$
(c) $-2 + 2i$ (d) $2 - 3i$

End Quiz

Solutions to Exercises

Exercise 1(a)

$$\begin{aligned}(3 + 2i) + (3 + i) &= 3 + 2i + 3 + i \\ &= 3 + 3 + 2i + 2i \\ &= 6 + 3i\end{aligned}$$

Click on the green square to return



Exercise 1(b) Here we need to be careful with the signs!

$$\begin{aligned}4 - 2i - (3 - 2i) &= 4 - 2i - 3 + 2i \\ &= 4 - 3 - 2i + 2i \\ &= 1\end{aligned}$$

A purely real result

Click on the green square to return



Exercise 1(c) The factor of $\frac{1}{2}$ multiplies both terms in the complex number.

$$\begin{aligned} -1 + 3i + \frac{1}{2}(2 + 2i) &= -1 + 3i + 1 + i \\ &= 4i \end{aligned}$$

A purely imaginary result.

[Click on the green square to return](#)



Exercise 1(d)

$$\begin{aligned}\frac{1}{3}(2 - 5i) - \frac{1}{6}(8 - 2i) &= \frac{2}{3} - \frac{5}{3}i - \frac{8}{6} + \frac{2}{6}i \\ &= \frac{2}{3} - \frac{5}{3}i - \frac{4}{3} + \frac{1}{3}i \\ &= \frac{2}{3} - \frac{4}{3} - \frac{5}{3}i + \frac{1}{3}i \\ &= -\frac{2}{3} - \frac{4}{3}i\end{aligned}$$

which we could also write as $-\frac{2}{3}(1 + 2i)$.

[Click on the green square to return](#)



Exercise 2(a)

$$\begin{aligned}(3 + 2i)(3 + i) &= 3 \times 3 + 3 \times i + 2i \times 3 + 2i \times i \\ &= 9 + 3i + 6i - 2 \\ &= 9 - 2 + 3i + 6i \\ &= 7 + 9i\end{aligned}$$

Click on the green square to return



Exercise 2(b)

$$\begin{aligned}(4 - 2i)(3 - 2i) &= 4 \times 3 + 4 \times (-2i) - 2i \times 3 - 2i \times -2i \\ &= 12 - 8i - 6i - 4 \\ &= 12 - 4 - 8i - 6i \\ &= 8 - 14i\end{aligned}$$

Click on the green square to return



Exercise 2(c)

$$\begin{aligned}(-1 + 3i)(2 + 2i) &= -1 \times 2 - 1 \times 2i + 3i \times 2 + 3i \times 2i \\ &= -2 - 2i + 6i - 6 \\ &= -2 - 6 - 2i + 6i \\ &= -8 + 4i\end{aligned}$$

Click on the green square to return



Exercise 2(d)

$$\begin{aligned}(2 - 5i)(8 - 3i) &= 2 \times 8 + 2 \times (-3i) - 5i \times 8 - 5i \times (-3i) \\ &= 16 - 6i - 40i - 15 \\ &= 16 - 15 - 6i - 40i \\ &= 1 - 46i\end{aligned}$$

Click on the green square to return



Exercise 3(a)

$$\begin{aligned}(3 + 2i) + (3 + 2i)^* &= (3 + 2i) + (3 - 2i) \\ &= 3 + 2i + 3 - 2i \\ &= 3 + 3 + 2i - 2i \\ &= 6\end{aligned}$$

Click on the green square to return



Exercise 3(b)

$$\begin{aligned}(3 - 2i)(3 - 2i)^* &= (3 - 2i)(3 + 2i) \\ &= 9 + 6i - 6i - 2i \times (2i) \\ &= 9 - 4i^2 \\ &= 9 + 4 = 13\end{aligned}$$

Click on the green square to return



Exercise 3(c)

$$\begin{aligned}(-1 + 3i)(-1 + 3i)^* &= (-1 + 3i)(-1 - 3i) \\ &= (-1) \times (-1) + (-1)(-3i) + 3i(-1) + 3i(-3i) \\ &= 1 + 3i - 3i - 9i^2 \\ &= 1 + 9 = 10\end{aligned}$$

Click on the green square to return



Exercise 3(d)

$$\begin{aligned}\sqrt{(4 - 3i)(4 + 3i)} &= \sqrt{4^2 + 4 \times 3i - 3i \times 4 - 3i \times 3i} \\ &= \sqrt{16 + 12i - 12i - 9i^2} \\ &= \sqrt{16 + 9} \\ &= \sqrt{25} = 5\end{aligned}$$

Click on the green square to return



Exercise 4(a)

$$\begin{aligned}\frac{(2 + 4i)}{i} &= \frac{(2 + 4i)}{i} \times \frac{-i}{-i} \\ &= \frac{(2 + 4i) \times (-i)}{+1} \\ &= (2 + 4i)(-i) \\ &= -2i - 4i^2 \\ &= 4 - 2i\end{aligned}$$

Click on the green square to return



Exercise 4(b)

$$\begin{aligned}\frac{(-2 + 6i)}{(1 + 2i)} &= \frac{(-2 + 6i)}{(1 + 2i)} \times \frac{(1 - 2i)}{(1 - 2i)} \\ &= \frac{(-2 + 6i)(1 - 2i)}{1 + 4} \\ &= \frac{1}{5}(-2 + 6i)(1 - 2i) \\ &= \frac{1}{5}(-2 + 4i + 6i - 12i^2) \\ &= \frac{1}{5}(-2 + 10i + 12) \\ &= \frac{1}{5}(10 + 10i) = 2 + 2i\end{aligned}$$

Click on the green square to return



Exercise 4(c)

$$\begin{aligned}\frac{(1+3i)}{(2+i)} &= \frac{(1+3i)}{(2+i)} \times \frac{(2-i)}{(2-i)} \\ &= \frac{(1+3i)(2-i)}{4+1} \\ &= \frac{1}{5}(2-i+6i-3i^2) \\ &= \frac{1}{5}(2+3+5i) \\ &= \frac{1}{5}(5+5i) = 1+i\end{aligned}$$

Click on the green square to return



Exercise 4(d)

$$\begin{aligned}\frac{(3+2i)}{(3+i)} &= \frac{(3+2i)}{(3+i)} \times \frac{(3-i)}{(3-i)} \\ &= \frac{(3+2i)(3-i)}{9+1} \\ &= \frac{1}{10}(3+2i)(3-i) \\ &= \frac{1}{10}(9-3i+6i-2i^2) \\ &= \frac{1}{10}(9+2+3i) \\ &= \frac{1}{10}(11+3i)\end{aligned}$$

Click on the green square to return



Solutions to Quizzes

Solution to Quiz:

$$\begin{aligned}(4 - 3i) + (2 + 5i) &= 4 - 3i + 2 + 5i \\ &= 4 + 2 - 3i + 5i \\ &= 6 + 2i\end{aligned}$$

End Quiz

Solution to Quiz:

Be careful with the signs!

$$\begin{aligned}(3 - i) - (2 - 6i) &= 3 - i - 2 + 6i \\ &= 3 - 2 - i + 6i \\ &= 1 + 5i\end{aligned}$$

End Quiz

Solution to Quiz:

$$\begin{aligned}(2 - i)(3 + 4i) &= 2 \times 3 + 2 \times (4i) - i \times 3 - i \times (4i) \\ &= 6 + 8i - 3i - 4i^2 \\ &= 6 + 5i + 4 \\ &= 10 + 5i\end{aligned}$$

End Quiz

Solution to Quiz:

$$\begin{aligned} |4 - 2i| &= \sqrt{(4 - 2i)(4 + 2i)} \\ &= \sqrt{4^2 + 2^2} \\ &= \sqrt{16 + 4} \\ &= \sqrt{20} \end{aligned}$$

End Quiz

Solution to Quiz:

$$\begin{aligned}\frac{8-i}{2+i} &= \frac{8-i}{2+i} \times \frac{2-i}{2-i} \\ &= \frac{(8-i)(2-i)}{2^2+1^2} \\ &= \frac{(8 \times 2 + 8 \times (-i) - i \times 2 - i \times (-i))}{5} \\ &= \frac{1}{5} (16 - 8i - 2i - 1) \\ &= \frac{1}{5} (15 - 10i) = 3 - 2i\end{aligned}$$

End Quiz

Solution to Quiz:

$$\begin{aligned}\frac{-2+i}{2+i} &= \frac{-2+i}{2+i} \frac{2-i}{2-i} \\ &= \frac{(-2+i)(2-i)}{2^2+1^2} \\ &= \frac{1}{5}(-2 \times 2 - 2 \times (-i) + i \times 2 + i \times (-i)) \\ &= \frac{1}{5}(-4 + 2i + 2i + 1) \\ &= \frac{1}{5}(-3 + 4i)\end{aligned}$$

End Quiz