Basic Mathematics

Factorising Expressions

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The aim of this document is to provide a short, self assessment programme for students who wish to acquire a basic competence at factorising simple algebraic expressions.

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Table of Contents

The full range of these packages and some instructions, should they be required, can be obtained from our web page Mathematics Support Materials.

1. Factorising Expressions (Introduction)

Expressions such as $(x + 5)(x - 2)$ were met in the package on brackets. There the emphasis was on the expansion of such expressions, which in this case would be $x^2 + 3x - 10$. There are many instances when the *reverse* of this procedure, i.e. factorising, is required. This section begins with some simple examples.

Example 1 Factorise the following expressions.

$$\text{(a) } 7x - x^2, \quad \text{(b) } 2abx + 2ab^2 + 2a^2b.$$

Solution

(a) This is easy since $7x - x^2 = x(7 - x)$.

(b) In this case the largest common factor is $2ab$ so

$$2abx + 2ab^2 + 2a^2b = 2ab(x + b + a).$$

On the next page are some exercises for you to try.

Exercise 1. Factorise each of the following expressions *as far as possible*. (Click on green letters for solutions.)

(a) $x^2 + 3x$
(b) $x^2 - 6x$
(c) $x^2y + y^3 + z^2y$
(d) $2ax^2y - 4ax^2z$
(e) $2a^3b + 5a^2b^2$
(f) $ayx + yx^3 - 2y^2x^2$

Quiz Which of the expressions below is the *full* factorisation of

$$16a - 2a^2\,?$$

(a) $a(16 - 2a)$
(b) $2(8 - 2a)$
(c) $2a(8 - a)$
(d) $2a(4 - 2a)$

Quiz Which of the following is the *full* factorisation of the expression

$$ab^2c - a^2bc^3 + 2abc^2\,?$$

(a) $abc(b - ac^2 + 2c)$
(b) $ab^2(c - ac^3 + ac)2$
(c) $ac(b^2 - abc^2 + 2bc)$
(d) $b^2c(a - abc^2 + ac)$

2. Further Expressions

Each of the previous expressions may be factored in a single operation. Many examples require more than one such operation. On the following page you will find some worked examples of this type.

Example 2 Factorise the expressions below *as far as possible*.

$$\text{(a)} \quad ax + ay + bx + by\,, \qquad \text{(b)} \quad 6ax - 3bx + 2ay - by\,.$$

Solution

(a) Note that a is a factor of the first two terms, and b is a factor of the second two. Thus

$$ax + ay + bx + by = a(x + y) + b(x + y)\,.$$

The expression in this form consists of a sum of two terms, each of which has the common factor $(x + y)$ so it may be further factorised. Thus

$$\begin{aligned} ax + ay + bx + by &= a(x + y) + b(x + y) \\ &= (a + b)(x + y)\,. \end{aligned}$$

(b) Here $3x$ is a factor of the first two terms and y is a factor of the second two. Thus

$$\begin{aligned} 6ax - 3bx + 2ay - by &= 3x(2a - b) + y(2a - b) \\ &= (3x + y)(2a - b)\,, \end{aligned}$$

taking out $(2a - b)$ as a common factor.

EXERCISE 2. Factorise each of the following *as fully as possible*. (Click on green letters for solution.)

(a) $xb + xc + yb + yc$ (b) $ah - ak + bh - bk$
(c) $hs + ht + ks + kt$ (d) $2mh - 2mk + nh - nk$
(e) $6ax + 2bx + 3ay + by$ (f) $ms + 2mt^2 - ns - 2nt^2$

Quiz Which of the following is the factorisation of the expression

$$2ax - 6ay - bx + 3by\,?$$

(a) $(2a + b)(x + 3y)$ (b) $(2a - b)(x - 3y)$
(c) $(2a + b)(x - 3y)$ (d) $(2a - b)(x + 3y)$

3. Quadratic Expressions

A *quadratic* expression is one of the form $ax^2 + bx + c$, with a, b, c being some *numbers*. When faced with a quadratic expression it is often, *but not always*, possible to *factorise it by inspection*. To get some insight into how this is done it is worthwhile looking at how such an expression is formed.

Suppose that a quadratic expression can be factored into two linear terms, say $(x + d)$ and $(x + e)$, where d, e are two *numbers*. Then the quadratic is

$$\begin{aligned}(x+d)(x+e) &= x^2 + xe + xd + de\,,\\ &= x^2 + (e+d)x + de\,,\\ &= x^2 + (d+e)x + de\,.\end{aligned}$$

Notice how it is formed. The coefficient of x is $(d + e)$, which is the *sum* of the two numbers in the linear terms $(x + d)$ and $(x + e)$. The final term, the one *without* an x, is the *product* of those two numbers. This is the information which is used to *factorise by inspection*.

Example 3 Factorise the following expressions.

$$\text{(a) } x^2 + 8x + 7\,, \quad \text{(b) } y^2 + 2y - 15\,.$$

Solution

(a) The only possible factors of 7 are 1 and 7, and these do add up to 8, so

$$x^2 + 8x + 7 = (x + 7)(x + 1)\,.$$

Checking this (see the package on Brackets for FOIL):

$$\begin{aligned}(x+7)(x+1) &= \overset{\mathbf{F}}{x^2} + \overset{\mathbf{O}}{x.1} + \overset{\mathbf{I}}{x.7} + \overset{\mathbf{L}}{7.1} \\ &= x^2 + 8x + 7\,.\end{aligned}$$

(b) Here the term independent of x (i.e. the one without an x) is *negative*, so the two numbers must be opposite in sign. The obvious contenders are 3 and −5, or −3 and 5. The first pair can be ruled out as their sum is −2. The second pair sum to +2, which is the correct coefficient for x. Thus

$$y^2 + 2y - 15 = (y - 3)(y + 5)\,.$$

Here are some examples for you to try.

EXERCISE 3. Factorise the following into *linear* factors. (Click on green letters for solution.)

(a) $x^2 + 7x + 10$
(b) $x^2 + 7x + 12$
(c) $y^2 + 11y + 24$
(d) $y^2 - 10y + 24$
(e) $z^2 - 3z - 10$
(f) $a^2 - 8a + 16$

Quiz Which of the following is the factorisation of the expression

$$z^2 - 6z + 8\,?$$

(a) $(z - 1)(z + 8)$
(b) $(z - 1)(z - 8)$
(c) $(z - 2)(z + 4)$
(d) $(z - 2)(z - 4)$

4. Quiz on Factorisation

Begin Quiz Factorise each of the following and choose the solution from the options given.

1. $2a^2e - 5ae^2 + a^3e^2$

(a) $ae(2a - 5e + a^2e)$ (b) $a^2e(2a - 5e + ae)$
(c) $ae(2a - 5e^2 + a^2e^2)$ (d) $a^2e(2 - 5e + a^2e^2)$

2. $6ax - 3bx + 2ay - by$

(a) $(3x - y)(2a + b)$ (b) $(3x + y)(2a - b)$
(c) $(3x - y)(2a - b)$ (d) $(3x + y)(2a + b)$

3. $z^2 - 26z + 165$

(a) $(z + 11)(z + 15)$ (b) $(z - 11)(z - 15)$
(c) $(z - 55)(z - 3)$ (d) $(z + 55)(z - 3)$

End Quiz Score: Correct

Solutions to Exercises

Exercise 1(a) The only common factor of the two terms is x so

$$x^2 + 3x = x(x + 3)\,.$$

Click on green square to return

□

Exercise 1(b) Again the two terms in the expression have only the common factor x, so

$$x^2 - 6x = x(x - 6)\,.$$

Click on green square to return

Exercise 1(c) Here the only common factor is y so

$$x^2y + y^3 + z^2y = y(x^2 + y^2 + z^2)\,.$$

Click on green square to return □

Exercise 1(d) In this case the largest common factor is $2ax^2$, so

$$2ax^2y - 4ax^2z = 2ax^2(y - 2z)\,.$$

Click on green square to return □

Exercise 1(e)

Here the largest common factor is a^2b, so this factorises as

$$2a^3b + 5a^2b^2 = a^2b(2a + 5b)\,.$$

Click on green square to return □

Exercise 1(f) The largest common factor is xy so

$$ayx + yx^3 - 2y^2x^2 = xy(a + x^2 - 2xy)\,.$$

Click on green square to return □

Exercise 2(a) We proceed as follows:

$$\begin{aligned} xb + xc + yb + yc &= x(b + c) + y(b + c) \\ &= (x + y)(b + c)\,. \end{aligned}$$

Click on green square to return

□

Exercise 2(b)

$$\begin{aligned} ah - ak + bh - bk &= a(h-k) + b(h-k) \\ &= (a+b)(h-k)\,. \end{aligned}$$

Click on green square to return

Exercise 2(c)

$$\begin{aligned} hs + ht + ks + kt &= h(s+t) + k(s+t) \\ &= (h+k)(s+t)\,. \end{aligned}$$

Click on green square to return □

Exercise 2(d)

$$\begin{aligned} 2mh - 2mk + nh - nk &= 2m(h-k) + n(h-k) \\ &= (2m+n)(h-k)\,. \end{aligned}$$

Click on green square to return □

Exercise 2(e)

$$\begin{aligned} 6ax + 2bx + 3ay + by &= 2x(3a+b) + y(3a+b) \\ &= (2x+y)(3a+b) \end{aligned}$$

Click on green square to return □

Exercise 2(f)

$$\begin{aligned} ms + 2mt^2 - ns - 2nt^2 &= m(s + 2t^2) - n(s + 2t^2) \\ &= (m - n)(s + 2t^2) \end{aligned}$$

Click on green square to return □

Exercise 3(a)
Since 10 has the factors 5 and 2, and their *sum* is 7,

$$\begin{aligned}(x+5)(x+2) &= x^2+2x+5x+10 \\ &= x^2+7x+10\,.\end{aligned}$$

Click on green square to return

Exercise 3(b)

Here there are several ways of factorising 12 but on closer inspection the only factors that work are 4 and 3. This leads to the following

$$\begin{aligned}(x+4)(x+3) &= x^2+3x+4x+12 \\ &= x^2+7x+12\,.\end{aligned}$$

Click on green square to return

□

Exercise 3(c)
There are several different possible factors for 24 but only one pair, 8 and 3 add up to 11. Thus

$$\begin{aligned}(y+8)(y+3) &= y^2+3y+8y+24\\ &= y^2+11y+24\,.\end{aligned}$$

Click on green square to return □

Exercise 3(d)

There are several different possible factors for 24 but only one pair, 6 and 4 add up to 10. Since the coefficient of y is negative, and the constant term is positive, the required numbers this time are -6 and -4. Thus

$$\begin{aligned}(y-6)(y-4) &= y^2 - 4y - 6y + (-6)(-4) \\ &= y^2 - 10y + 24\,.\end{aligned}$$

Click on green square to return □

Exercise 3(e) The constant term in this case is negative. Since this is the *product* of the numbers required, they must have *opposite* signs, i.e. one is positive and one negative. In that case, the number in front of the x must be the *difference* of these two numbers. On inspection, 5 and 2 have product 10 and difference 3. Since the x term is negative, the larger number must be negative.

$$\begin{aligned}(z-5)(z+2) &= z^2+2z-5z+(-5\times 2)\\ &= z^2-3z-10\,.\end{aligned}$$

Click on green square to return □

Exercise 3(f)

This is an example of a perfect square. These are mentioned in the package on Brackets. The factors of 16 in this case are -4 and -4.

$$\begin{aligned} (a-4)^2 &= (a-4)(a-4) \\ &= a^2 - 4a - 4a + (-4)\times(-4) \\ &= a^2 - 8a + 16\,. \end{aligned}$$

Click on green square to return ☐

Solutions to Quizzes

Solution to Quiz: Here 2 is a factor of both terms, but so is a, so the *largest common factor* is $2a$. Thus

$$16a - 2a^2 = 2a(8 - a)\,.$$

End Quiz

Solution to Quiz:

The largest common factor in this case is $a \times b \times c = abc$. Thus

$$\begin{aligned} ab^2c - a^2bc^3 + 2abc^2 &= (abc \times b) - (abc \times ac^2) + (abc \times 2c) \\ &= abc(b - ac^2 + 2c) \end{aligned}$$

End Quiz

Solution to Quiz: Noting that $2a$ is a factor of the first two terms and $-b$ is a factor of the second two, we have

$$\begin{aligned} 2ax - 6ay - bx + 3by &= 2a(x - 3y) - b(x - 3y) \\ &= (2a - b)(x - 3y) \end{aligned}$$

End Quiz

Solution to Quiz: Here the two numbers have product 8, so a possible choice is 2 and 4. However their sum in this case is 6, whereas the sum required is -6. Taking the pair to be -2 and -4 will give the same product, $+8$, but with the correct sum. Thus

$$z^2 - 6z + 8 = (z - 4)(z - 2)\,,$$

and this can be checked by expanding the brackets.

End Quiz