



Gradients and Directional Derivatives

R Horan & M Lavelle

The aim of this package is to provide a short self assessment programme for students who want to obtain an ability in vector calculus to calculate gradients and directional derivatives.

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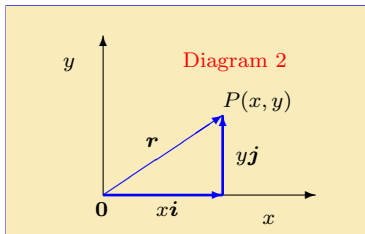
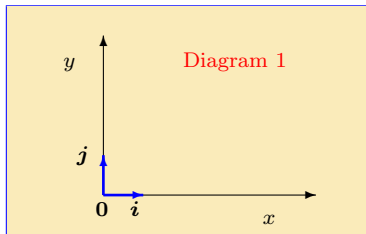
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The full range of these packages and some instructions, should they be required, can be obtained from our web page [Mathematics Support Materials](#).

1. Introduction (Vectors)

The **base vectors** in two dimensional Cartesian coordinates are the unit vector \mathbf{i} in the positive direction of the x axis and the unit vector \mathbf{j} in the y direction. See **Diagram 1**. (In three dimensions we also require \mathbf{k} , the unit vector in the z direction.)

The **position vector** of a point $P(x, y)$ in two dimensions is $x\mathbf{i} + y\mathbf{j}$. We will often denote this important vector by \mathbf{r} . See **Diagram 2**. (In three dimensions the position vector is $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.)



The **vector differential operator** ∇ , called “del” or “nabla”, is **defined** in three dimensions to be:

$$\nabla = \frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + \frac{\partial}{\partial z} \mathbf{k}.$$

Note that these are *partial derivatives*!

This vector operator may be applied to (differentiable) scalar functions (scalar fields) and the result is a special case of a vector field, called a gradient vector field.

Here are two warming up exercises on partial differentiation.

Quiz Select the following partial derivative, $\frac{\partial}{\partial z} (xyz^x)$.

- (a) x^2yz^{x-1} , (b) 0 , (c) $xy \log_x(z)$, (d) yz^{x-1} .

Quiz Choose the partial derivative $\frac{\partial}{\partial x} (x \cos(y) + y)$.

- (a) $\cos(y)$, (b) $\cos(y) - x \sin(y) + 1$,
(c) $\cos(y) + x \sin(y) + 1$, (d) $-\sin(y)$.

2. Gradient (Grad)

The **gradient** of a function, $f(x, y)$, in two dimensions is defined as:

$$\text{grad}f(x, y) = \nabla f(x, y) = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j}.$$

The **gradient** of a function is a **vector field**. It is obtained by applying the vector operator ∇ to the scalar function $f(x, y)$. Such a vector field is called a **gradient (or conservative) vector field**.

Example 1 The **gradient** of the function $f(x, y) = x + y^2$ is given by:

$$\begin{aligned} \nabla f(x, y) &= \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} \\ &= \frac{\partial}{\partial x}(x + y^2) \mathbf{i} + \frac{\partial}{\partial y}(x + y^2) \mathbf{j} \\ &= (1 + 0) \mathbf{i} + (0 + 2y) \mathbf{j} \\ &= \mathbf{i} + 2y \mathbf{j}. \end{aligned}$$

Quiz Choose the gradient of $f(x, y) = x^2y^3$.

(a) $2xi + 3y^2j$,

(b) $x^2i + y^3j$,

(c) $2xy^3i + 3x^2y^2j$,

(d) $y^3i + x^2j$.

The definition of the **gradient** may be extended to functions defined in three dimensions, $f(x, y, z)$:

$$\nabla f(x, y) = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k}.$$

EXERCISE 1. Calculate the **gradient** of the following functions (click on the **green** letters for the solutions).

(a) $f(x, y) = x + 3y^2$,

(b) $f(x, y) = \sqrt{x^2 + y^2}$,

(c) $f(x, y, z) = 3x^2\sqrt{y} + \cos(3z)$,

(d) $f(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$,

(e) $f(x, y) = \frac{4y}{(x^2 + 1)}$,

(f) $f(x, y, z) = \sin(x)e^y \ln(z)$.

3. Directional Derivatives

To interpret the gradient of a scalar field

$$\nabla f(x, y, z) = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k},$$

note that its component in the \mathbf{i} direction is the partial derivative of f with respect to x . This is the rate of change of f in the x direction since y and z are kept constant. In general, **the component of ∇f in any direction is the rate of change of f in that direction.**

Example 2 Consider the scalar field $f(x, y) = 3x + 3$ in two dimensions. It has no y dependence and it is linear in x . Its gradient is given by

$$\begin{aligned} \nabla f &= \frac{\partial}{\partial x}(3x + 3)\mathbf{i} + \frac{\partial}{\partial y}(3x + 3)\mathbf{j} \\ &= 3\mathbf{i} + 0\mathbf{j}. \end{aligned}$$

As would be expected the gradient has zero component in the y direction and its component in the x direction is constant (3).

Quiz Select a point from the answers below at which the scalar field $f(x, y, z) = x^2yz - xy^2z$ decreases in the y direction.

- (a) $(1, -1, 2)$, (b) $(1, 1, 1)$,
(c) $(-1, 1, 2)$, (d) $(1, 0, 1)$.

Definition: if \hat{n} is a unit vector, then $\hat{n} \cdot \nabla f$ is called the **directional derivative** of f in the direction \hat{n} . The directional derivative is the rate of change of f in the direction \hat{n} .

Example 3 Let us find the directional derivative of $f(x, y, z) = x^2yz$ in the direction $4\mathbf{i} - 3\mathbf{k}$ at the point $(1, -1, 1)$.

The vector $4\mathbf{i} - 3\mathbf{k}$ has magnitude $\sqrt{4^2 + (-3)^2} = \sqrt{25} = 5$. The **unit vector** in the direction $4\mathbf{i} - 3\mathbf{k}$ is thus $\hat{n} = \frac{1}{5}(4\mathbf{i} - 3\mathbf{k})$.

The gradient of f is

$$\begin{aligned}\nabla f &= \frac{\partial}{\partial x}(x^2yz)\mathbf{i} + \frac{\partial}{\partial y}(x^2yz)\mathbf{j} + \frac{\partial}{\partial z}(x^2yz)\mathbf{k} \\ &= 2xyz\mathbf{i} + x^2z\mathbf{j} + x^2y\mathbf{k},\end{aligned}$$

and so the required directional derivative is

$$\begin{aligned}\hat{\mathbf{n}} \cdot \nabla f &= \frac{1}{5}(4\mathbf{i} - 3\mathbf{k}) \cdot (2xyz\mathbf{i} + x^2z\mathbf{j} + x^2y\mathbf{k}) \\ &= \frac{1}{5} [4 \times 2xyz + 0 - 3 \times x^2y].\end{aligned}$$

At the point $(1, -1, 1)$ the desired directional derivative is thus

$$\hat{\mathbf{n}} \cdot \nabla f = \frac{1}{5} [8 \times (-1) - 3 \times (-1)] = -1.$$

EXERCISE 2. Calculate the **directional derivative** of the following functions in the given **directions** and at the stated **points** (click on the **green** letters for the solutions).

(a) $f = 3x^2 - 3y^2$ in the direction \mathbf{j} at $(1, 2, 3)$.

(b) $f = \sqrt{x^2 + y^2}$ in the direction $2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ at $(0, -2, 1)$.

(c) $f = \sin(x) + \cos(y) + \sin(z)$ in the direction $\pi\mathbf{i} + \pi\mathbf{j}$ at $(\pi, 0, \pi)$.

We now state, without proof, **two useful properties** of the directional derivative and gradient.

- The maximal directional derivative of the scalar field $f(x, y, z)$ is in the direction of the gradient vector ∇f .
- If a surface is given by $f(x, y, z) = c$ where c is a constant, then the normals to the surface are the vectors $\pm \nabla f$.

Example 4 Consider the surface $xy^3 = z + 2$. To find its unit normal at $(1, 1, -1)$, we need to write it as $f = xy^3 - z = 2$ and calculate the gradient of f :

$$\nabla f = y^3 \mathbf{i} + 3xy^2 \mathbf{j} - \mathbf{k}.$$

At the point $(1, 1, -1)$ this is $\nabla f = \mathbf{i} + 3\mathbf{j} - \mathbf{k}$. The *magnitude* of this maximal rate of change is

$$\sqrt{1^2 + 3^2 + (-1)^2} = \sqrt{11}.$$

Thus the unit normals to the surface are $\pm \frac{1}{\sqrt{11}}(\mathbf{i} + 3\mathbf{j} - \mathbf{k})$.

Quiz Which of the following vectors is normal to the surface $x^2yz = 1$ at $(1,1,1)$?

(a) $4\mathbf{i} + \mathbf{j} + 17\mathbf{k}$,

(b) $2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$,

(c) $\mathbf{i} + \mathbf{j} + \mathbf{k}$,

(d) $-2\mathbf{i} - \mathbf{j} - \mathbf{k}$.

Quiz Which of the following vectors is a unit normal to the surface $\cos(x)yz = -1$ at $(\pi,1,1)$?

(a) $-\frac{1}{\sqrt{2}}\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k}$,

(b) $\pi\mathbf{i} + \mathbf{j} + \frac{2}{\sqrt{\pi}}\mathbf{k}$,

(c) \mathbf{i} ,

(d) $-\frac{1}{\sqrt{2}}\mathbf{j} - \frac{1}{\sqrt{2}}\mathbf{k}$.

Quiz Select a unit normal to the (spherically symmetric) surface $x^2 + y^2 + z^2 = 169$ at $(5,0,12)$.

(a) $\mathbf{i} + \frac{1}{6}\mathbf{j} - \frac{1}{6}\mathbf{k}$,

(b) $\frac{1}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}$,

(c) $\frac{5}{13}\mathbf{i} + \frac{12}{13}\mathbf{k}$,

(d) $-\frac{5}{13}\mathbf{i} + \frac{12}{13}\mathbf{k}$.

4. Final Quiz

Begin Quiz Choose the solutions from the options given.

- What is the gradient of $f(x, y, z) = xyz^{-1}$?
(a) $\mathbf{i} + \mathbf{j} - z^{-2}\mathbf{k}$, (b) $\frac{y}{z}\mathbf{i} + \frac{x}{z}\mathbf{j} - \frac{xy}{z^2}\mathbf{k}$,
(c) $yz^{-1}\mathbf{i} + xz^{-1}\mathbf{j} + xyz^{-2}\mathbf{k}$, (d) $-\frac{1}{z^2}$.
- If n is a constant, choose the gradient of $f(\mathbf{r}) = 1/r^n$, where $r = |\mathbf{r}|$ and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.
(a) 0, (b) $-\frac{n}{2} \frac{\mathbf{i} + \mathbf{j} + \mathbf{k}}{r^{n+1}}$, (c) $-\frac{n\mathbf{r}}{r^{n+2}}$, (d) $-\frac{n}{2} \frac{\mathbf{r}}{r^{n+2}}$.
- Select the unit normals to the surface of revolution, $z = 2x^2 + 2y^2$ at the point (1,1,4).
(a) $\pm \frac{1}{\sqrt{3}}(\mathbf{i} + \mathbf{j} - \mathbf{k})$, (b) $\pm \frac{1}{\sqrt{3}}(\mathbf{i} + \mathbf{j} + \mathbf{k})$,
(c) $\pm \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j})$, (d) $\pm \frac{1}{\sqrt{2}}(2\mathbf{i} + 2\mathbf{j} - 4\mathbf{k})$.

End Quiz

Solutions to Exercises

Exercise 1(a) The function $f(x, y) = x + 3y^2$, has gradient

$$\begin{aligned}\nabla f(x, y) &= \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} \\ &= \frac{\partial}{\partial x} (x + 3y^2) \mathbf{i} + \frac{\partial}{\partial y} (x + 3y^2) \mathbf{j} \\ &= (1 + 0) \mathbf{i} + (0 + 3 \times 2y^{2-1}) \mathbf{j} \\ &= \mathbf{i} + 6y \mathbf{j} .\end{aligned}$$

Click on the **green** square to return



Exercise 1(b) The gradient of the function

$$f(x, y) = \sqrt{x^2 + y^2} = (x^2 + y^2)^{\frac{1}{2}}$$

is given by:

$$\begin{aligned}\nabla f(x, y) &= \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} = \frac{\partial}{\partial x} (x^2 + y^2)^{\frac{1}{2}} \mathbf{i} + \frac{\partial}{\partial y} (x^2 + y^2)^{\frac{1}{2}} \mathbf{j} \\ &= \frac{1}{2} (x^2 + y^2)^{\frac{1}{2}-1} \times \frac{\partial}{\partial x} (x^2) \mathbf{i} \\ &\quad + \frac{1}{2} (x^2 + y^2)^{\frac{1}{2}-1} \times \frac{\partial}{\partial y} (y^2) \mathbf{j} \\ &= \frac{1}{2} (x^2 + y^2)^{-\frac{1}{2}} \times 2x^{2-1} \mathbf{i} + \frac{1}{2} (x^2 + y^2)^{-\frac{1}{2}} \times 2y^{2-1} \mathbf{j} \\ &= (x^2 + y^2)^{-\frac{1}{2}} x \mathbf{i} + (x^2 + y^2)^{-\frac{1}{2}} y \mathbf{j} \\ &= \frac{x}{\sqrt{x^2 + y^2}} \mathbf{i} + \frac{y}{\sqrt{x^2 + y^2}} \mathbf{j}.\end{aligned}$$

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Exercise 1(c) The **gradient** of the function

$$f(x, y, z) = 3x^2\sqrt{y} + \cos(3z) = 3x^2y^{\frac{1}{2}} + \cos(3z),$$

is given by:

$$\begin{aligned}\nabla f(x, y, z) &= \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k} \\ &= 3y^{\frac{1}{2}} \frac{\partial}{\partial x} (x^2) \mathbf{i} + 3x^2 \frac{\partial}{\partial y} (y^{\frac{1}{2}}) \mathbf{j} + \frac{\partial}{\partial z} (\cos(3z)) \mathbf{k} \\ &= 3y^{\frac{1}{2}} \times 2x^{2-1} \mathbf{i} + 3x^2 \times \frac{1}{2} y^{\frac{1}{2}-1} \mathbf{j} - 3 \sin(3z) \mathbf{k} \\ &= 6y^{\frac{1}{2}} x \mathbf{i} + \frac{3}{2} x^2 y^{-\frac{1}{2}} \mathbf{j} - 3 \sin(3z) \mathbf{k} \\ &= 6x\sqrt{y} \mathbf{i} + \frac{3}{2} \frac{x^2}{\sqrt{y}} \mathbf{j} - 3 \sin(3z) \mathbf{k}.\end{aligned}$$

Click on the **green** square to return



Exercise 1(d) The partial derivative of the function

$$f(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}} = (x^2 + y^2 + z^2)^{-\frac{1}{2}},$$

with respect to the variable x is

$$\frac{\partial f}{\partial x} = -\frac{1}{2}(x^2 + y^2 + z^2)^{-\frac{1}{2}-1} \times \frac{\partial(x^2)}{\partial x} = -\frac{x}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}$$

and similarly the derivatives $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$ are

$$\frac{\partial f}{\partial y} = -\frac{y}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}, \quad \frac{\partial f}{\partial z} = -\frac{z}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}.$$

Therefore the **gradient** is

$$\nabla f(x, y, z) = -\frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}.$$

Click on the **green** square to return



Exercise 1(e) The **gradient** of the function

$$f(x, y) = \frac{4y}{(x^2 + 1)} = 4y(x^2 + 1)^{-1},$$

is:

$$\begin{aligned}\nabla f(x, y) &= 4y \times \frac{\partial}{\partial x}(x^2 + 1)^{-1} \mathbf{i} + (x^2 + 1)^{-1} \times \frac{\partial}{\partial y} 4y \mathbf{j} \\ &= 4y \times (-1)(x^2 + 1)^{-1-1} \frac{\partial}{\partial x}(x^2 + 1) \mathbf{i} + 4(x^2 + 1)^{-1} \mathbf{j} \\ &= -4y(x^2 + 1)^{-2} \times 2x \mathbf{i} + \frac{4}{(x^2 + 1)} \mathbf{j} \\ &= -\frac{8xy}{(x^2 + 1)^2} \mathbf{i} + \frac{4}{(x^2 + 1)} \mathbf{j}.\end{aligned}$$

Click on the **green** square to return



Exercise 1(f) The partial derivatives of the function

$$f(x, y, z) = \sin(x)e^y \ln(z)$$

are

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} (\sin(x)) e^y \ln(z) = \cos(x) e^y \ln(z),$$

$$\frac{\partial f}{\partial y} = \sin(x) \frac{\partial}{\partial y} (e^y) \ln(z) = \sin(x) e^y \ln(z),$$

$$\frac{\partial f}{\partial z} = \sin(x) e^y \frac{\partial}{\partial z} (\ln(z)) = \sin(x) e^y \frac{1}{z}.$$

Therefore the gradient is

$$\nabla f(x, y, z) = \cos(x) e^y \ln(z) \mathbf{i} + \sin(x) e^y \ln(z) \mathbf{j} + \sin(x) e^y \frac{1}{z} \mathbf{k}.$$

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Exercise 2(a) The directional derivative of the function

$$f = 3x^2 - 3y^2$$

in the unit vector \mathbf{j} direction is given by the scalar product $\mathbf{j} \cdot \nabla f$.

The gradient of the function $f = 3x^2 - 3y^2$ is

$$\nabla f = 6xi - 6yj$$

Therefore the directional derivative in the \mathbf{j} direction is

$$\mathbf{j} \cdot \nabla f = \mathbf{j} \cdot (6xi - 6yj) = -6y$$

and at the point $(1, 2, 3)$ it has the value $-6 \times 2 = -12$.

Click on the **green** square to return



Exercise 2(b) The directional derivative of the function $f = \sqrt{x^2 + y^2}$ in the direction defined by vector $2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ is given by the scalar product $\hat{\mathbf{n}} \cdot \nabla f$, where the unit vector $\hat{\mathbf{n}}$ is

$$\hat{\mathbf{n}} = \frac{2\mathbf{i} + 2\mathbf{j} + \mathbf{k}}{\sqrt{2^2 + 2^2 + 1^2}} = \frac{2\mathbf{i} + 2\mathbf{j} + \mathbf{k}}{\sqrt{9}} = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}.$$

The gradient of the function f is

$$\nabla f = \frac{x}{\sqrt{x^2 + y^2}}\mathbf{i} + \frac{y}{\sqrt{x^2 + y^2}}\mathbf{j} + 0\mathbf{k} = \frac{x\mathbf{i} + y\mathbf{j}}{\sqrt{x^2 + y^2}}$$

Therefore the required directional derivative is

$$\hat{\mathbf{n}} \cdot \nabla f = \left(\frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k} \right) \cdot \left(\frac{x\mathbf{i} + y\mathbf{j}}{\sqrt{x^2 + y^2}} \right) = \frac{2}{3} \frac{x + y}{\sqrt{x^2 + y^2}}.$$

At the point $(0, -2, 1)$ it is equal to $\frac{2}{3} \frac{0 - 2}{\sqrt{0^2 + (-2)^2}} = \frac{2}{3} \times \frac{-2}{2} = -\frac{2}{3}$.

Click on the **green** square to return



Exercise 2(c) The directional derivative of the function

$$f = \sin(x) + \cos(x) + \sin(z)$$

in the direction defined by the vector $\pi\mathbf{i} + \pi\mathbf{j}$ is given by the scalar product $\hat{\mathbf{n}} \cdot \nabla f$, where the unit vector $\hat{\mathbf{n}}$ is

$$\hat{\mathbf{n}} = \frac{\pi\mathbf{i} + \pi\mathbf{j}}{\sqrt{\pi^2 + \pi^2}} = \frac{\mathbf{i} + \mathbf{j}}{\sqrt{2}}.$$

The gradient of the function f is

$$\nabla f = \cos(x)\mathbf{i} - \sin(y)\mathbf{j} + \cos(z)\mathbf{k}.$$

Therefore the directional derivative is

$$\hat{\mathbf{n}} \cdot \nabla f = \frac{\mathbf{i} + \mathbf{j}}{\sqrt{2}} \cdot (\cos(x)\mathbf{i} - \sin(y)\mathbf{j} + \cos(z)\mathbf{k}) = \frac{\cos(x) - \sin(y)}{\sqrt{2}}$$

and at the point $(\pi, 0, \pi)$ it becomes $\frac{\cos(\pi) - \sin(0)}{\sqrt{2}} = -\frac{1}{\sqrt{2}}$.

Click on the **green** square to return



Solutions to Quizzes

Solution to Quiz:

The partial derivative of xyz^x with respect to the variable z is

$$\frac{\partial}{\partial z} (xyz^x) = xy \times \frac{\partial}{\partial z} (z^x) = xy \times x \times z^{(x-1)} = x^2 y z^{(x-1)}$$

End Quiz

Solution to Quiz:

Consider the function $f(x, y) = x \cos(y) + y$, its derivative with respect to the variable x is

$$\begin{aligned}\frac{\partial}{\partial x} f(x, y) &= \frac{\partial}{\partial x} (x \cos(y) + y) \\ &= \frac{\partial}{\partial x} (x) \times \cos(y) + \frac{\partial}{\partial x} (y) \\ &= 1 \times \cos(y) + 0 = \cos(y).\end{aligned}$$

End Quiz

Solution to Quiz:

The **gradient** of the function $f(x, y) = x^2y^3$ is given by:

$$\begin{aligned}\nabla f(x, y) &= \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} \\ &= \frac{\partial}{\partial x}(x^2y^3) \mathbf{i} + \frac{\partial}{\partial y}(x^2y^3) \mathbf{j} \\ &= \frac{\partial}{\partial x}(x^2) \times y^3 \mathbf{i} + x^2 \times \frac{\partial}{\partial y}(y^3) \mathbf{j} \\ &= 2x^{2-1} \times y^3 \mathbf{i} + x^2 \times 3y^{3-1} \mathbf{j} \\ &= 2xy^3 \mathbf{i} + 3x^2y^2 \mathbf{j} .\end{aligned}$$

End Quiz

Solution to Quiz: The partial derivative of the scalar function $f(x, y, z) = x^2yz - xy^2z$ with respect to y is

$$\frac{\partial f}{\partial y}(x, y, z) = x^2 - 2xyz.$$

Evaluating it at the point $(1, 1, 1)$ gives

$$\frac{\partial f}{\partial y}(1, 1, 1) = 1^2 - 2 \times 1 \times 1 \times 1 = 1 - 2 = -1.$$

This is negative and therefore the function f decreases in the y direction at this point.

It may be verified that the function does not decrease in the y direction at any of the other three points.

End Quiz

Solution to Quiz: The surface is defined by the equation

$$x^2yz = 1.$$

To find its normal at $(1, 1, 1)$ we need to calculate the gradient of the function $f(x, y, z) = x^2yz$:

$$\nabla f = 2xyz\mathbf{i} + x^2z\mathbf{j} + x^2y\mathbf{k}.$$

At the point $(1, 1, 1)$ this is

$$\nabla f = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$$

Thus the required normals to the surface are $\pm(2\mathbf{i} + \mathbf{j} + \mathbf{k})$. Hence (d) is a normal vector to the surface. End Quiz

Solution to Quiz: The surface is defined by the equation

$$\cos(x)yz = -1.$$

To find its unit normal at the point $(\pi, 1, 1)$, we need to evaluate the gradient of $f(x, y, z) = \cos(x)yz$:

$$\nabla f = -\sin(x)yz\mathbf{i} + \cos(x)z\mathbf{j} + \cos(x)y\mathbf{k}.$$

At the point $(\pi, 1, 1)$ this is

$$\nabla f = 0\mathbf{i} + (-1)\mathbf{j} + (-1)\mathbf{k} = -\mathbf{j} - \mathbf{k}$$

The magnitude of this vector is

$$\sqrt{(-1)^2 + (-1)^2} = \sqrt{2}.$$

Therefore the unit normal is

$$\hat{\mathbf{n}} = -\frac{1}{\sqrt{2}}\mathbf{j} - \frac{1}{\sqrt{2}}\mathbf{k}.$$

End Quiz

Solution to Quiz: The surface is defined by the equation

$$x^2 + y^2 + z^2 = 169.$$

To find its unit normal at point $(5, 0, 12)$ we need to evaluate the gradient of $f(x, y, z) = x^2 + y^2 + z^2$:

$$\nabla f = 2xi + 2yj + 2zk.$$

At the point $(5, 0, 12)$ this is

$$\nabla f = 2 \times 5i + 0 \times j + 2 \times 12k = 10i + 24k$$

The magnitude of this vector is

$$\sqrt{(2 \times 5)^2 + (2 \times 12)^2} = \sqrt{4 \times (25 + 144)} = 2\sqrt{169} = 2 \times 13.$$

Therefore the unit normal is

$$\hat{n} = \frac{5}{13}j + \frac{12}{13}k.$$

End Quiz