Basic Physics

The Lens Equation

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The aim of this package is to provide a short self assessment programme for students who want to understand image formation by lenses.

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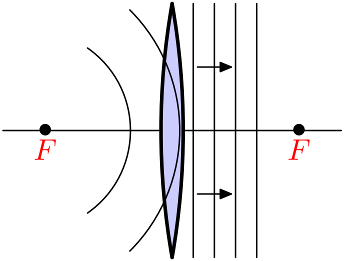
Table of Contents

The full range of these packages and some instructions, should they be required, can be obtained from our web page Mathematics Support Materials.

1. Introduction (Refraction and Lenses)

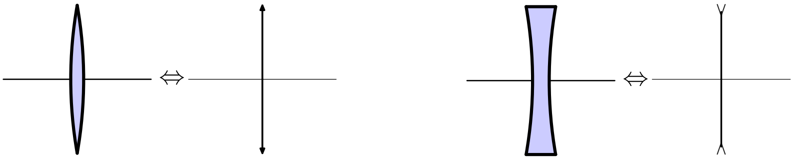
Lenses use **refraction** to produce images of objects. Optical instruments, such as telescopes or cameras, and your eyes work this way.

The surface of the lens in the diagram below is shaped so that spherical wave fronts of light coming from a point, F, called the **focal point**, emerge as parallel wave fronts (so-called plane waves).

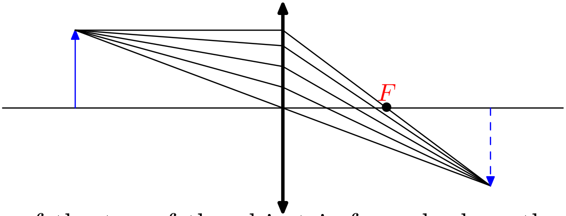


Note that the line drawn above through the middle of the lens is called the **principal axis**.

Convex (converging) and concave (diverging) lenses are drawn as



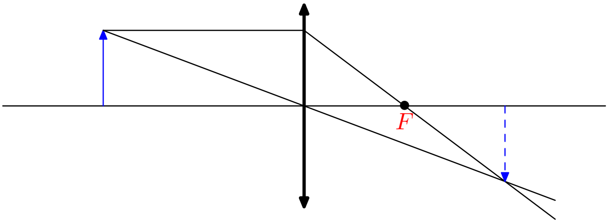
To understand image formation we use **ray diagrams**. Here is an example for a convex lens:



The image of the top of the object is formed where the light rays cross. In a perfect lens all the rays from a point on the object will meet at one other point - so we only need to draw two rays!

For a convex lens, we draw the ray diagram as follows:

- Draw a ray from the top of the object straight through the middle of the lens. Its direction is not changed.
- Draw a ray from the top of the object parallel to the principal axis. It is refracted by the lens to pass through the focal point.



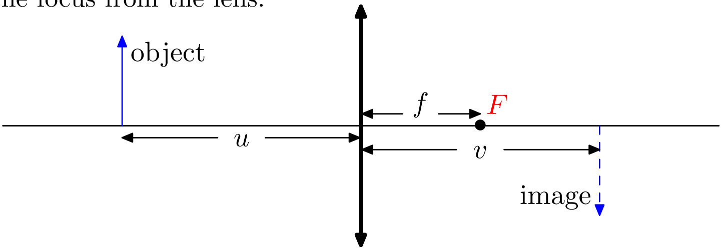
From the diagram we see that the image in this example is **inverted**. This is also an example of a **real image** as the light rays pass through the image's location and may be seen on a screen placed there.

2. The Lens Equation

An image formed by a convex lens is described by the **lens equation**

$$\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$$

where u is the distance of the object from the lens; v is the distance of the image from the lens and f is the *focal length*, i.e., the distance of the focus from the lens.



N.B. other sign conventions are sometimes used in the literature.

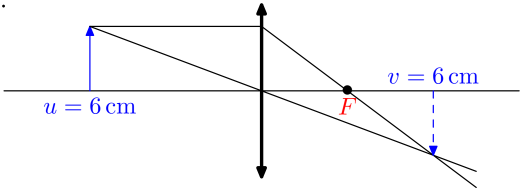
Example 1 What image is produced by placing an object 6 cm away from a convex lens of focal length 3 cm?
The question states that $u = 6\,\text{cm}$ and $f = 3\,\text{cm}$. This can be substituted into the lens equation as follows:

$$\frac{1}{u} + \frac{1}{v} = \frac{1}{f} \qquad \therefore \frac{1}{6} + \frac{1}{v} = \frac{1}{3}$$

$$\frac{1}{v} = \frac{1}{3} - \frac{1}{6}$$

$$= \frac{2}{6} - \frac{1}{6} = \frac{1}{6}$$

So $v = 6\,\text{cm}$. From the **ray diagram** we see that this is an **inverted, real image**.



Here are two quizzes similar to the above example.

Quiz If an object is 12 cm away from a convex lens of focal length 4 cm where will the image be? (Draw a ray diagram of how this image is formed to see if your answer is plausible.)

(a) $v = 6\,\text{cm}$, (b) $v = 8\,\text{cm}$, (c) $v = 16\,\text{cm}$, (d) $v = 3\,\text{cm}$.

Quiz If an object is 8 cm away from a convex lens of focal length 2 cm where will the image be? (Draw a ray diagram of how this image is formed to see if your answer is plausible.)

(a) $v = \frac{8}{5}\,\text{cm}$, (b) $v = 6\,\text{cm}$, (c) $v = 10\,\text{cm}$, (d) $v = \frac{8}{3}\,\text{cm}$.

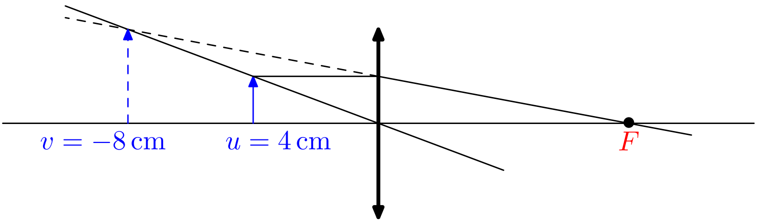
Example 2 What image is produced by placing an object $4\,\mathrm{cm}$ away from a convex lens of focal length $8\,\mathrm{cm}$?

The question states that $u = 4\,\mathrm{cm}$ and $f = 8\,\mathrm{cm}$. This can be substituted into the lens equation as follows:

$$\begin{aligned} \frac{1}{u} + \frac{1}{v} &= \frac{1}{f} \\ \therefore \frac{1}{4} + \frac{1}{v} &= \frac{1}{8} \\ \frac{1}{v} &= \frac{1}{8} - \frac{2}{8} \\ \frac{1}{v} &= -\frac{1}{8} \end{aligned}$$

This implies that $v = -8\,\mathrm{cm}$. What does this *negative* result mean? The **ray diagram** on the next page clarifies the nature of the image.

The ray diagram is drawn using the two rules from p. 5.



We see that the image is on the same side of the lens as the object! This is the significance of v, the image position, being negative. The image is also **upright**.

It is also evident that the light rays only *appear* to pass through the position of the image. No image would be cast on a screen placed at the image location. This is called a **virtual image**. You can also see a virtual image by looking in a mirror on a wall – your image will appear to be behind the wall!

Exercise 1. From the lens equation calculate the position of the following images produced by a convex lens. Draw a ray diagram in each case and state whether the images are real or virtual, and also if they are upright or inverted (click on the **green** letters for the solutions).

(a) $u = 6\,\text{cm}\,, \quad f = 2\,\text{cm}$ (b) $u = 6\,\text{cm}\,, \quad f = 4\,\text{cm}$

(c) $u = 10\,\text{mm}\,, \quad f = 15\,\text{mm}$ (d) $u = 90\,\text{mm}\,, \quad f = 15\,\text{mm}$

Quiz Which of the following combinations has an upright image $12\,\text{cm}$ away from a convex lens?

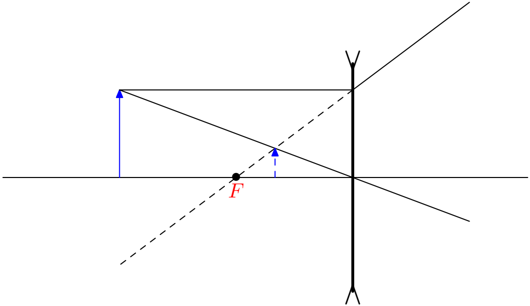
(a) $u = 4\,\text{cm}\,, \quad f = 3\,\text{cm}$ (b) $u = 13\,\text{cm}\,, \quad f = 1\,\text{cm}$

(c) $u = 3\,\text{cm}\,, \quad f = 4\,\text{cm}$ (d) $u = 6\,\text{cm}\,, \quad f = 6\,\text{cm}$

3. Concave Lenses

Concave lenses always produce *upright, virtual images*. For a concave lens, the lens equation is the same but the value of f is now negative. Ray diagrams for such lenses are drawn using:

- a ray from the top of the object through the middle of the lens;
- a ray from the top of the object parallel to the principal axis which the lens refracts so it seems to come from the focal point.

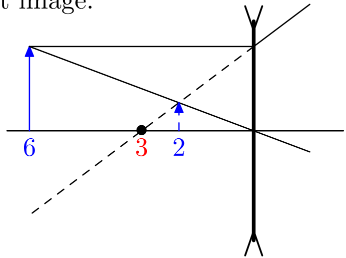


Example 3 If an object is 6 cm from a concave lens with focal length $f = -3\,\text{cm}$, what is its position and nature?
From the lens equation

$$\begin{aligned} \frac{1}{u} + \frac{1}{v} &= \frac{1}{f} \\ \therefore \frac{1}{6} + \frac{1}{v} &= -\frac{1}{3} \\ \frac{1}{v} &= -\frac{2}{6} - \frac{1}{6} = -\frac{1}{2} \end{aligned}$$

So $v = -2\,\text{cm}$. This is a virtual, upright image.

Concave lenses always create virtual images *between* the object and the lens.



Here are some exercises with concave lenses. Be *careful with the sign* of the focal length in the lens formula!

Exercise 2. Calculate the position of the images formed by the following **concave lenses**. (click on the **green** letters for the solutions).

(a) For an object with $u = 12\,\text{cm}$ if the focal length is $-4\,\text{cm}$

(b) For an object with $u = 6\,\text{cm}$ if the focal length is $-2\,\text{cm}$

(c) For an object with $u = 4\,\text{cm}$ if the focal length is $-8\,\text{cm}$

(d) For an object with $u = 240\,\text{mm}$ if the focal length is $-8\,\text{cm}$

In the next section we will see how knowing the positions of the object and image enables us to work out the magnification of the image.

4. Magnification

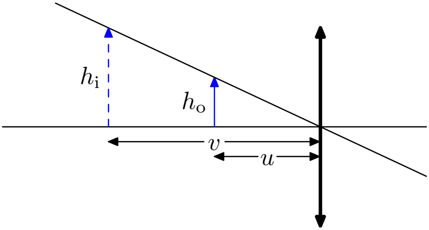
The magnification M of an image is the ratio of the height of the image to the height of the object:

$$M = \frac{\text{image height}}{\text{object height}}$$

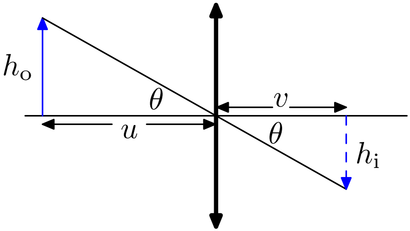
This number is a dimensionless ratio (a length over a length) and does not have any units. To calculate M for virtual images, consider the diagram below:

From similar triangles:

$$\begin{aligned} \frac{h_\text{i}}{v} &= \frac{h_\text{o}}{u} \\ \Rightarrow \frac{h_\text{i}}{vh_\text{o}} &= \frac{1}{u} \\ \therefore \frac{h_\text{i}}{h_\text{o}} &= \frac{v}{u} \end{aligned}$$



This result also holds for real images:



From the diagram, using similar triangles:

$$\frac{h_o}{u} = \frac{h_i}{v}$$

and just as before:

$$M = \frac{h_i}{h_o} = \frac{v}{u}$$

i.e., exactly the same equations as before.

Rule: The magnification factor M of a lens is always positive and given by:

$$M = \frac{|v|}{u}$$

Example 4 If an object is $8\,\text{cm}$ from a lens and produces an image $4\,\text{cm}$ from the lens, what is the magnification factor, M?

$$M = \frac{v}{u} = \frac{4}{8} = \frac{1}{2}$$

Thus the image will be half the size of the object.

EXERCISE 3. Calculate the magnification factor M or size of image for the following cases. (click on the **green** letters for the solutions).

(a) M if $u = 6\,\text{cm}$ and $v = 24\,\text{cm}$

(b) M if $u = 6\,\text{cm}$ and $v = -12\,\text{cm}$

(c) The image height h_{i} of an object with height $2\,\text{cm}$, if $u = 3\,\text{cm}$ and $v = -9\,\text{cm}$

5. Final Quiz

Begin Quiz Choose the solutions from the options given.

1. For a convex lens of focal length 3 cm, where will the image of an object 12 cm in front of the lens appear?
(a) -9 cm (b) 4 cm (c) 6 cm (d) 3 cm

2. Which of the following properties does **not** apply to the above image?
(a) the image is real (b) the image is inverted
(c) the image is magnified by 4 (d) the image is magnified by $\frac{1}{3}$

3. Select from the choices below the object position which, for a convex lens with $f = 130$ mm, will lead to a virtual image
(a) 18 cm (b) 0.05 m (c) 300 mm (d) 260 mm

4. Select v, if $u = 6$ cm for a concave lens with $f = 4$ cm.
(a) -2 cm (b) -24 mm (c) -1 cm (d) -12 mm

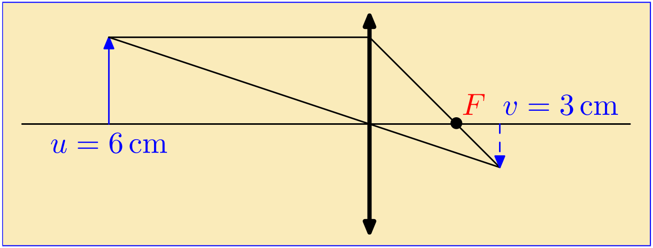
End Quiz Score: Correct

Solutions to Exercises

Exercise 1(a) The object is $u = 6\,\text{cm}$ away from a convex lens of focal length $f = 2\,\text{cm}$. The image position can be found from the lens equation

$$\frac{1}{6} + \frac{1}{v} = \frac{1}{2}, \quad \therefore \quad \frac{1}{v} = \frac{1}{2} - \frac{1}{6} = \frac{3}{6} - \frac{1}{6} = \frac{1}{3}.$$

This implies that $v = 3\,\text{cm}$. The image is real and inverted.

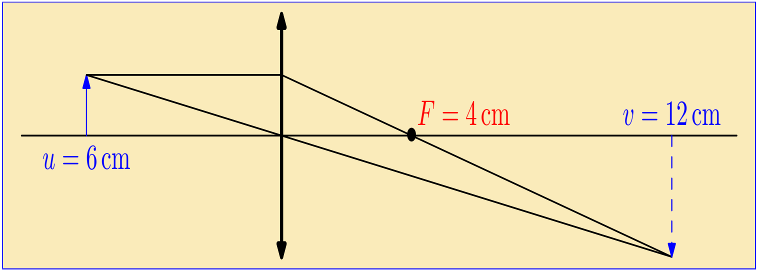


Click on the **green** square to return

Exercise 1(b) The object is again at the same distance $u = 6\,\text{cm}$, but the convex lens has focal length $f = 4\,\text{cm}$. The position of the image may be calculated from the lens equation

$$\frac{1}{6} + \frac{1}{v} = \frac{1}{4}\,, \quad \therefore \quad \frac{1}{v} = \frac{1}{4} - \frac{1}{6} = \frac{3}{12} - \frac{2}{12} = \frac{1}{12}\,.$$

This yields $v = 12\,\text{cm}$. From the diagram, this is real, inverted image.



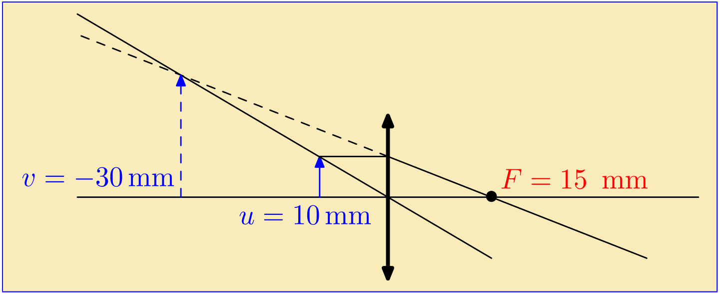
Click on the **green** square to return

□

Exercise 1(c) If the object is at a distance $u = 10\,\mathrm{mm}$ away from a convex lens of focal length $f = 15\,\mathrm{mm}$, then the position v of the image can be found as follows:

$$\frac{1}{10} + \frac{1}{v} = \frac{1}{15}\,, \quad \therefore \frac{1}{v} = \frac{1}{15} - \frac{1}{10} = \frac{2}{30} - \frac{3}{30} = -\,\frac{1}{30}\,.$$

The negative sign shows that the image is on the same side of the lens as the object, $v = -30\,\mathrm{mm}$. This is an upright, virtual image.

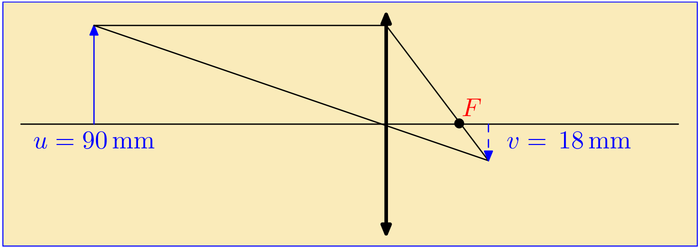


Click on the **green** square to return □

Exercise 1(d) The object is at a distance $u = 90\,\mathrm{mm}$ away from a convex lens of focal length $f = 15\,\mathrm{mm}$. The position v of the image is calculated from the lens equation via

$$\frac{1}{90} + \frac{1}{v} = \frac{1}{15}\,, \quad \therefore \quad \frac{1}{v} = \frac{1}{15} - \frac{1}{90} = \frac{6}{90} - \frac{1}{90} = \frac{1}{18}\,.$$

The image appears at the same distance $v = 18\,\mathrm{mm}$ as in Exercise 1 (c), but now behind the lens, i.e., it is real and inverted.

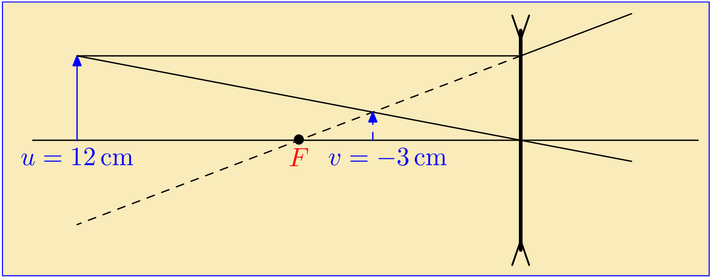


Click on the **green** square to return ☐

Exercise 2(a) Consider an object at a distance $u = 12\,\text{cm}$ away from a concave lens of focal length $f = -4\,\text{cm}$. The position v of the image can be calculated from the lens equation:

$$\frac{1}{12} + \frac{1}{v} = -\frac{1}{4}\,, \quad \therefore \quad \frac{1}{v} = -\frac{1}{4} - \frac{1}{12} = -\frac{3}{12} - \frac{1}{12} = -\frac{1}{3}\,.$$

So, an upright image appears at $v = -3\,\text{cm}$ from the lens on the same as the object.



Click on the **green** square to return

□

Exercise 2(b) If an object is at a distance $u = 6\,\mathrm{cm}$ away from a concave lens of focal length $f = -2\,\mathrm{cm}$, then the position v of the image can be found as follows

$$\begin{aligned} \frac{1}{6} + \frac{1}{v} &= -\frac{1}{2}\,, \\ \frac{1}{v} &= -\frac{1}{2} - \frac{1}{6} = -\frac{3}{6} - \frac{1}{6} = -\,\frac{2}{3}\,. \end{aligned}$$

So, a virtual upright image appears at $v = -\frac{3}{2}\,\mathrm{cm}$.

Click on the **green** square to return

□

Exercise 2(c) When an object is at a distance $u = 4\,\text{cm}$ away from a concave lens of focal length $f = -8\,\text{cm}$, then the position v of the image may be evaluated from the lens equation as follows

$$\begin{aligned} \frac{1}{4} + \frac{1}{v} &= -\frac{1}{8}\,, \\ \frac{1}{v} &= -\frac{1}{8} - \frac{1}{4} = -\frac{1}{8} - \frac{2}{8} = -\frac{3}{8}\,. \end{aligned}$$

A virtual upright image appears at $v = -\frac{8}{3}\,\text{cm}$.

Click on the **green** square to return ☐

Exercise 2(d) Consider an object placed at a distance $u = 240\,\text{mm} = 24\,\text{cm}$ away from a concave lens of focal length $f = -8\,\text{cm}$, then the position v of the image is determined from the lens equation

$$\begin{aligned}\frac{1}{24} + \frac{1}{v} &= -\frac{1}{8}\,,\\ \frac{1}{v} &= -\frac{1}{8} - \frac{1}{24} = -\frac{3}{24} - \frac{1}{24} = -\frac{1}{6}\,.\end{aligned}$$

Therefore, an upright image appears on the same side as the object at the distance $v = -6\,\text{cm}$.

Click on the **green** square to return □

Exercise 3(a) When an object placed at $u = 6\,\text{cm}$ away from a lens produces an image at $v = 24\,\text{cm}$ from the lens then the magnification factor M is

$$M = \frac{v}{u} = \frac{24}{6} = 4\,.$$

This means that the image is four times the size of the object.

Click on the **green** square to return

□

Exercise 3(b) For an object at $u = 6\,\mathrm{cm}$ away from a lens and its image at $v = -12\,\mathrm{cm}$ from the lens the magnification factor M is

$$M = \frac{|v|}{u} = \frac{12}{6} = 2\,.$$

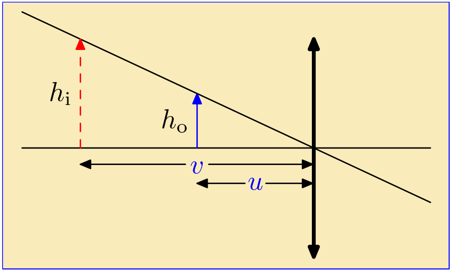
Thus the image is twice the size of the object.

Click on the **green** square to return

□

Exercise 3(c)

Consider an object with height $h_o = 2\,\text{cm}$ placed at $u = 3\,\text{cm}$ away from the lens. If its image is at $v = -9\,\text{cm}$, then from the diagram, using similar triangles, it follows that



$$\frac{h_o}{u} = \frac{h_i}{v}, \qquad \therefore \quad \frac{2}{3} = \frac{h_i}{9}.$$

From this equation one can find the size of the image

$$h_i = 9 \times \frac{2}{3} = 6\,\text{cm}.$$

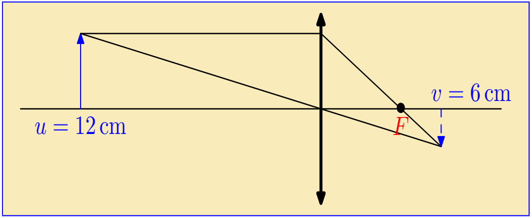
Click on the **green** square to return □

Solutions to Quizzes

Solution to Quiz: From the question the distance of the object from the lens of focal length $f = 4\,\text{cm}$ is $u = 12\,\text{cm}$. Substituting this data into the lens equation gives:

$$\begin{aligned} \frac{1}{12} + \frac{1}{v} &= \frac{1}{4}, \\ \frac{1}{v} &= \frac{1}{4} - \frac{1}{12} = \frac{3}{12} - \frac{1}{12} = \frac{1}{6}. \end{aligned}$$

So $v = 6\,\text{cm}$. This is drawn below.

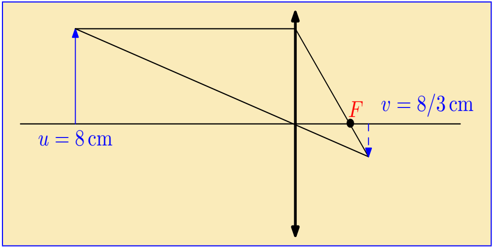


End Quiz

Solution to Quiz: According to the question the distance of an object from the lens of focal length $f = 2\,\text{cm}$ is $u = 8\,\text{cm}$. Plugging these numbers into the lens equation gives:

$$\begin{aligned} \frac{1}{8} + \frac{1}{v} &= \frac{1}{2}, \\ \frac{1}{v} &= \frac{1}{2} - \frac{1}{8} = \frac{4}{8} - \frac{1}{8} = \frac{3}{8}. \end{aligned}$$

So $v = \dfrac{8}{3}\,\text{cm}$. From the diagram,this is a real,inverted image.



End Quiz

Solution to Quiz: Consider an object at a distance $u = 3\,\mathrm{cm}$ away from a convex lens of focal length $f = 4\,\mathrm{cm}$. The position v of the image is calculated from the lens equation

$$\begin{aligned} \frac{1}{3} + \frac{1}{v} &= \frac{1}{4}\,, \\ \therefore \quad \frac{1}{v} &= \frac{1}{4} - \frac{1}{3} = \frac{3}{12} - \frac{4}{12} = -\,\frac{1}{12}\,. \end{aligned}$$

So, an upright image appears at a distance $v = -12\,\mathrm{cm}$ on the same side of the lens as the object.

End Quiz