



Maxima and Minima

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The aim of this document is to provide a short, self assessment programme for students who wish to be able to use differentiation to find maxima and minima of functions.

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1. Rules of Differentiation

Throughout this package the following rules of differentiation will be assumed. (In the table of derivatives below, a is an arbitrary, non-zero constant.)

y	ax^n	$\sin(ax)$	$\cos(ax)$	e^{ax}	$\ln(ax)$
$\frac{dy}{dx}$	nax^{n-1}	$a \cos(ax)$	$-a \sin(ax)$	ae^{ax}	$\frac{1}{x}$

If a is any constant and u, v are two functions of x , then

$$\begin{aligned}\frac{d}{dx}(u + v) &= \frac{du}{dx} + \frac{dv}{dx} \\ \frac{d}{dx}(au) &= a \frac{du}{dx}\end{aligned}$$

The *stationary points* of a function are those points where the gradient of the tangent (the derivative of the function) is zero.

Example 1 Find the stationary points of the functions

$$(a) f(x) = 3x^2 + 2x - 9, \quad (b) f(x) = x^3 - 6x^2 + 9x - 2.$$

Solution

$$(a) \text{ If } y = 3x^2 + 2x - 9 \text{ then } \frac{dy}{dx} = 3 \times 2x^{2-1} + 2 = 6x + 2.$$

The *stationary points* are found by solving the equation

$$\frac{dy}{dx} = 6x + 2 = 0.$$

In this case there is only one solution, $x = -1/3$. Substituting this into the equation for $f(x)$, the corresponding y value is then

$$\begin{aligned} y &= 3(-1/3)^2 + 2(-1/3) - 9 \\ &= -28/3. \end{aligned}$$

The stationary point is thus the point with coordinates $(-1/3, -28/3)$.

(b) If $y = f(x) = x^3 - 6x^2 + 9x - 2$, then $\frac{dy}{dx} = 3x^2 - 12x + 9$.

The **stationary points** are found by solving $3x^2 - 12x + 9 = 0$. This is a quadratic equation (see the package on **quadratic equations**) and may be solved by factorising.

$$\begin{aligned}0 = 3x^2 - 12x + 9 &= 3(x^2 - 4x + 3) \\ &= 3(x - 3)(x - 1).\end{aligned}$$

There are two solutions in this case, $x = 3$ and $x = 1$. If these are substituted into the function, the two stationary points will be found to be $(3, -2)$ and $(1, 2)$.

EXERCISE 1. Find the *stationary points* of the following functions. (Click on the **green** letters for solutions.)

(a) $y = 16 - 6u - u^2$,

(b) $y = 3x^2 - 4x + 7$,

(c) $y = \frac{1}{3}x^3 - x^2 - 3x + 2$,

(d) $y = x^3 - 6x^2 - 15x + 16$.

2. Derivatives of order 2 (and higher)

In the package on **introductory differentiation** the height, $h(t)$, in metres, of a ball thrown vertically at 10 ms^{-1} , was given by

$$h(t) = 10t - 5t^2.$$

The *velocity* of the ball, $v \text{ ms}^{-1}$, after t seconds, was given by

$$v(t) = \frac{dh}{dt} = 10 - 10t.$$

The rate of change of velocity with time, which is the *acceleration*, is then given by $a(t)$, where

$$a(t) = \frac{dv}{dt} = -10 \text{ ms}^{-1}.$$

Since the acceleration was derived from $h(t)$ by two successive differentiations, the resulting function, which in this case is $a(t)$, is called the *second derivative* of $h(t)$ with respect to t . In symbols

$$a(t) = \frac{dv}{dt} = \frac{d}{dt} \left(\frac{dh}{dt} \right) = \frac{d^2h}{dt^2}. \quad \left(\begin{array}{l} \text{Note the position} \\ \text{of the superscripts} \end{array} \right)$$

Example 2

Find the *first*, *second* and *third* derivatives of the function

$$y = 3x^6 - 2x + 1.$$

Solution

The *first* derivative is given by

$$\frac{dy}{dx} = 18x^5 - 2.$$

The *second* derivative is given by

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = 90x^4.$$

The *third* derivative is given by

$$\frac{d^3y}{dx^3} = \frac{d}{dx} \left(\frac{d^2y}{dx^2} \right) = 360x^3.$$

EXERCISE 2. Find the *first*, *second* and *third* derivatives of the following functions. (Click on the **green** letters for solutions.)

(a) $y = x^3 + e^{2x}$,

(b) $y = \sin(x)$.

3. Maxima and Minima

The diagram below shows part of a function $y = f(x)$.

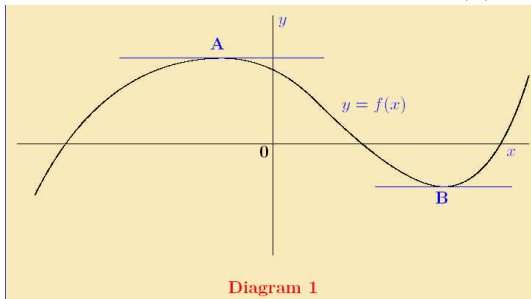
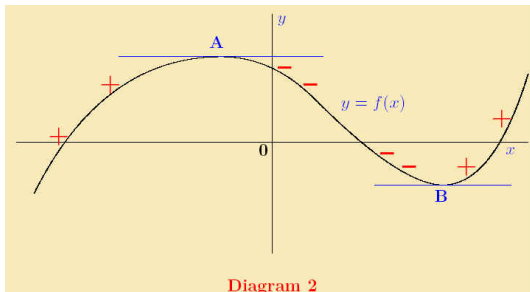


Diagram 1

The point A is a *local maximum* and the point B is a *local minimum*. At each of these points the tangent to the curve is parallel to the x -axis so the derivative of the function is zero. Both of these points are therefore *stationary points* of the function. The term *local* is used since these points are the maximum and minimum in this particular region. There may be others outside this region.



Further observations on the gradients of tangents to the curve are:

- to the *left* of **A** the gradients are positive (+)
- between **A** and **B** the gradients are negative (-)
- to the *right* of **B** the gradients are positive (+)

About the *local maximum* point **A** the gradient changes from positive, to zero, to negative. The gradient is therefore *decreasing*.

About the *local minimum* point **B** the gradient changes from negative, to zero, to positive. The gradient is therefore *increasing*.

The rate of change of a function is measured by its derivative.

When the derivative is *positive*, the function is *increasing*,

when the derivative is *negative*, the function is *decreasing*.

Thus the rate of change of the *gradient* is measured by *its derivative*, which is the *second derivative* of the original function. In mathematical notation this is as follows.

At the point (a, b)

if $\frac{dy}{dx} = 0$
and $\frac{d^2y}{dx^2} < 0$
then the point (a, b) is a
local maximum.

if $\frac{dy}{dx} = 0$
and $\frac{d^2y}{dx^2} > 0$
then the point (a, b) is a
local minimum.

Example 3

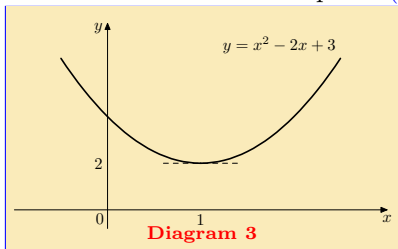
Find the stationary point of the function $y = x^2 - 2x + 3$ and hence determine the nature of this point.

Solution

If $y = x^2 - 2x + 3$ then $\frac{dy}{dx} = 2x - 2$ and $\frac{d^2y}{dx^2} = 2$.

Now $\frac{dy}{dx} = 2x - 2 = 0$ when $x = 1$. The function has only one stationary point when $x = 1$ (and $y = 2$). Since $\frac{d^2y}{dx^2} = 2 > 0$ for all values of x , this stationary point is a *local minimum*. Thus the function $y = x^2 - 2x + 3$ has a *local minimum* at the point $(1, 2)$.

The graph shows the function $y = x^2 - 2x + 3$ with the local minimum point at $(1, 2)$ clearly visible.



EXERCISE 3. Determine the nature of the *stationary points* of the following functions. (Click on the **green** letters for solutions.)

(a) $y = 16 - 6u - u^2$, (b) $y = 3x^2 - 4x + 7$,

Quiz Which of the following points is a local maximum of the function $y = 2x^3 - 15x^2 - 36x + 6$?

(a) $(-3, 15)$, (b) $(-6, 25)$, (c) $(-2, 25)$, (d) $(-1, 25)$.

Example 4

Find the stationary points of the function $y = 2x^3 - 9x^2 + 12x - 3$ and determine their nature.

Solution

If $y = 2x^3 - 9x^2 + 12x - 3$

then $\frac{dy}{dx} = 6x^2 - 18x + 12$

and $\frac{d^2y}{dx^2} = 12x - 18$.

The stationary points are found by solving

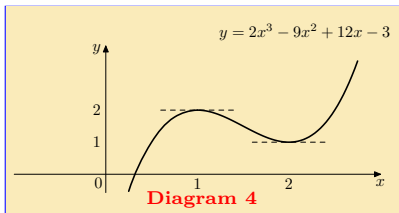
$$\frac{dy}{dx} = 6x^2 - 18x + 12 = 0.$$

Now $6x^2 - 18x + 12 = 6(x^2 - 3x + 2)$ so the solution is found by solving $x^2 - 3x + 2 = 0$, i.e. $x = 1$ and $x = 2$. (See the package on **quadratics**.) **Note** that when $x = 1$, $y = 2$ and when $x = 2$, $y = 1$.

When $x = 1$, $\frac{d^2y}{dx^2} = 12(1) - 18 = -6 < 0$, so $(1, 2)$ is a *local maximum*.

When $x = 2$, $\frac{d^2y}{dx^2} = 12(2) - 18 = +6 > 0$, so $(2, 1)$ is a *local minimum*.

The graph shows the function $y = 2x^3 - 9x^2 + 12x - 3$ with the *local maximum* point at $(1, 2)$ and the *local minimum* point at $(2, 1)$ clearly visible.



EXERCISE 4. Determine the nature of the *stationary points* of the following functions. (Click on the **green** letters for solutions.)

(a) $y = \frac{1}{3}x^3 - x^2 - 3x + 2$, (b) $y = \frac{1}{3}s^2 - 12s + 32$,

(c) $y = x^3 - 6x^2 - 15x + 16$, (d) $y = x^3 - 12x + 12$.

Quiz Referring to the function $y = x + 1 + \frac{1}{x}$, which of the following statements is true?

- (a) $(1, 3)$ is a *local maximum*, (b) $(1, -3)$ is a *local minimum*,
(c) $(-1, 1)$ is a *local minimum*, (d) $(-1, -1)$ is a *local maximum*.

Quiz Referring to the function $y = \frac{x^2}{2} - \cos x$, which of the following statements is true?

- (a) $(\frac{\pi}{2}, \frac{\pi^2}{8})$ is a *local minimum*, (b) $(\frac{\pi}{2}, \frac{\pi^2}{8})$ is a *local maximum*,
(c) $(0, -1)$ is a *local minimum*, (d) $(0, -1)$ is a *local maximum*.

4. Quiz on Max and Min

Begin Quiz In each of the following cases, choose the correct option.

- If $y = 3x^2 + 6x + 2$ then
 - $(1, 11)$ is a *local minimum*,
 - $(-1, -1)$ is a *local minimum*,
 - $(1, 11)$ is a *local maximum*,
 - $(-1, -1)$ is a *local maximum*.
- The function $y = \frac{x^3}{3} + \frac{x^2}{2} - 2x + 4$ has a
 - local minimum* when $x = 2$,
 - local maximum* when $x = -2$,
 - local maximum* when $x = 1$,
 - local minimum* when $x = -1$.
- The function $y = \sin x + \cos x$ has a
 - local maximum* when $x = \frac{\pi}{4}$,
 - local minimum* when $x = \frac{\pi}{4}$,
 - local maximum* when $x = \frac{\pi}{2}$,
 - local minimum* when $x = \frac{\pi}{2}$.

End Quiz

Solutions to Exercises

Exercise 1(a) If $y = f(u) = 16 - 6u - u^2$, then its derivative is

$$\frac{dy}{du} = -6 - 2u.$$

The stationary points are found by solving the linear equation

$$-6 - 2u = 0.$$

In this case there is only one solution $u = -\frac{6}{2} = -3$. Substituting this into the expression for the function $f(u) = 16 - 6u - u^2$, we find the corresponding y value

$$y = 16 - 6 \times (-3) - (-3)^2 = 16 + 18 - 9 = 25.$$

The stationary point is thus the point with coordinates $(-3, 25)$.

[Click on the green square to return](#)



Exercise 1(b) If $y = f(x) = 3x^2 - 4x + 7$, then its derivative is

$$\frac{dy}{dx} = 6x - 4.$$

The stationary points are found by solving the linear equation

$$6x - 4 = 0.$$

In this case there is only one solution $x = \frac{4}{6} = \frac{2}{3}$. Substituting this into the expression for the function $f(x) = 3x^2 - 4x + 7$, we find the corresponding y value

$$y = 3 \times \left(\frac{2}{3}\right)^2 - 4 \times \frac{2}{3} + 7 = \frac{4}{3} - \frac{8}{3} + 7 = \frac{17}{3}.$$

Therefore the function $y = 3x^2 - 4x + 7$ has only one stationary point $\left(\frac{2}{3}, \frac{17}{3}\right)$.

[Click on the green square to return](#)



Exercise 1(c) If $y = f(x) = \frac{1}{3}x^3 - x^2 - 3x + 2$, then $\frac{dy}{dx} = x^2 - 2x - 3$.

The stationary points are found by solving the quadratic equation $x^2 - 2x - 3 = 0$. Factorisation of this equation gives

$$0 = x^2 - 2x - 3 = (x + 1)(x - 3).$$

yielding two solutions, $x = -1$ and $x = 3$. Substituting $x = -1$ into the function $f(x)$, the corresponding y value is

$$y = \frac{1}{3}(-1)^3 - (-1)^2 - 3(-1) + 2 = -\frac{1}{3} - 1 + 3 + 2 = \frac{11}{3}.$$

Similarly substituting $x = 3$

$$y = \frac{1}{3}(3)^3 - (3)^2 - 3(3) + 2 = 9 - 9 - 9 + 2 = -7.$$

The two stationary points are thus $(-1, \frac{11}{3})$ and $(3, -7)$.

[Click on the green square to return](#)



Exercise 1(d) If $y = f(x) = x^3 - 6x^2 - 15x + 16$, then

$$\frac{dy}{dx} = 3x^2 - 12x - 15.$$

The stationary points are found by solving the quadratic equation $3x^2 - 12x - 15 = 0$. Factorising it

$$0 = 3x^2 - 12x - 15 = 3(x^2 - 4x - 5) = 3(x + 1)(x - 5)$$

we obtain two solutions, $x = -1$ and $x = 5$. Substituting $x = -1$ into the function $f(x)$, the corresponding y value is

$$y = (-1)^3 - 6 \times (-1)^2 - 15 \times (-1) + 16 = -1 - 6 + 15 + 16 = 24.$$

Similarly substituting $x = 5$

$$y = (5)^3 - 6 \times (5)^2 - 15 \times (5) + 16 = 125 - 6 \times 25 - 15 \times 5 + 16 = -84.$$

Two stationary points are thus $(-1, 24)$ and $(5, -84)$.

Click on the green square to return



Exercise 2(a) If $y = x^3 + e^{2x}$ then the *first* derivative is given by

$$\frac{dy}{dx} = 3 \times x^{(3-1)} + 2 \times e^{2x} = 3x^2 + 2e^{2x} .$$

The *second* derivative is given by

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = 2 \times 3x^{(2-1)} + 2 \times 2e^{2x} = 6x + 4e^{2x} .$$

The *third* derivative is given by

$$\frac{d^3y}{dx^3} = \frac{d}{dx} \left(\frac{d^2y}{dx^2} \right) = 6 + 8e^{2x} .$$

Click on the green square to return



Exercise 2(b) If $y = \sin(x)$ then the *first* derivative is given by

$$\frac{dy}{dx} = \cos(x).$$

The *second* derivative is given by

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = -\sin(x),$$

The *third* derivative is given by

$$\frac{d^3y}{dx^3} = \frac{d}{dx} \left(\frac{d^2y}{dx^2} \right) = -\cos(x).$$

Click on the green square to return



Exercise 3(a) If $y = 16 - 6u - u^2$ then

$$\frac{dy}{du} = -6 - 2u \quad \text{and} \quad \frac{d^2y}{du^2} = -2.$$

There is one stationary point when $u = -3$ which is obtained from the equation

$$\frac{dy}{du} = -6 - 2u = 0.$$

At this point $y = 16 - 6 \times (-3) - (-3)^2 = 25$. Since $\frac{d^2y}{du^2} = -2 < 0$ for all values of x , this stationary point is a *local maximum*. Thus the function $y = 16 - 6u - u^2$ has a *local maximum* at the point $(-3, 25)$.

Click on the green square to return



Exercise 3(b) If $y = 3x^2 - 4x + 7$ then

$$\frac{dy}{dx} = 6x - 4 \quad \text{and} \quad \frac{d^2y}{dx^2} = 6.$$

There is one stationary point $x = \frac{2}{3}$ which is the solution to the equation

$$\frac{dy}{dx} = 6x - 4 = 0.$$

At this point $y = 3 \times \left(\frac{2}{3}\right)^2 - 4 \times \left(\frac{2}{3}\right) + 7 = \frac{17}{3}$. Since $\frac{d^2y}{dx^2} = 6 > 0$ for all values of x , this stationary point is a *local minimum*. Thus the function $y = 3x^2 - 4x + 7$ has a *local minimum* at the point $\left(\frac{2}{3}, \frac{17}{3}\right)$.

Click on the green square to return



Exercise 4(a) If $y = f(x) = \frac{1}{3}x^3 - x^2 - 3x + 2$ then

$$\frac{dy}{dx} = x^2 - 2x - 3 \quad \text{and} \quad \frac{d^2y}{dx^2} = 2x - 2.$$

There are two stationary points at $x = -1$ and $x = 3$ which are the two solutions to the quadratic equation

$$\frac{dy}{dx} = x^2 - 2x - 3 = (x + 1)(x - 3) = 0.$$

When $x = -1$, $y = \frac{11}{3}$ and its second derivative has the value

$\frac{d^2y}{dx^2} = 2 \times (-1) - 2 = -4 < 0$. The point $(-1, \frac{11}{3})$ is a *local maximum*.

At the second point $x = 3$ and $y = -7$. The second derivative has the value $\frac{d^2y}{dx^2} = 2 \times (3) - 2 = 4 > 0$ and the point $(3, -7)$ is a *local minimum*.

Click on the green square to return



Exercise 4(b) If $y = f(s) = \frac{1}{3}s^2 - 12s + 32$ then

$$\frac{dy}{ds} = \frac{2}{3}s - 12 \quad \text{and} \quad \frac{d^2y}{ds^2} = \frac{2}{3}.$$

There is only one stationary point, when $s = 18$, which is obtained from the equation

$$\frac{dy}{ds} = \frac{2}{3}s - 12 = \frac{2}{3}(s - 18) = 0.$$

When $s = 18$ the function $f(s)$ takes the value $y = -76$. Since its second derivative $\frac{d^2y}{ds^2} = \frac{2}{3}$ is positive for all values of s , the stationary point $(18, -76)$ is a *local minimum*.

Click on the green square to return



Exercise 4(c) If $y = f(x) = x^3 - 6x^2 - 15x + 16$ then

$$\frac{dy}{dx} = 3x^2 - 12x - 15 \quad \text{and} \quad \frac{d^2y}{dx^2} = 6x - 12.$$

There are two stationary points $x = -1$ and $x = 5$ which are obtained from solving the quadratic equation

$$\frac{dy}{dx} = 3x^2 - 12x - 15 = 3(x^2 - 4x - 5) = 3(x + 1)(x - 5) = 0.$$

When $x = -1$ the function value is $y = 24$ and its second derivative is $\frac{d^2y}{dx^2} = 6 \times (-1) - 12 = -18 < 0$. The point $(-1, 24)$ is a *local maximum*.

At the second point, $x = 5$, the function value is $y = -84$ and the second derivative is $\frac{d^2y}{dx^2} = 6 \times (5) - 12 = 18 > 0$. The point $(5, -84)$ is a *local minimum*.

Click on the green square to return



Exercise 4(d) If $y = f(x) = x^3 - 12x + 12$ then

$$\frac{dy}{dx} = 3x^2 - 12 \quad \text{and} \quad \frac{d^2y}{dx^2} = 6x.$$

There are two stationary points $x = -2$ and $x = 2$ which are obtained from the equation

$$\frac{dy}{dx} = 3x^2 - 12 = 3(x^2 - 4) = 3(x + 2)(x - 2) = 0.$$

At the point $x = -2$ the function value is $y = 28$ and its second derivative is $\frac{d^2y}{dx^2} = 6 \times (-2) = -12 < 0$. The point $(-2, 28)$ is therefore a *local maximum*.

At the second point $x = 2$ the function value is $y = -4$ and the second derivative is $\frac{d^2y}{dx^2} = 6 \times (2) = 12 > 0$. The point $(2, -4)$ is a *local minimum*.

Click on the green square to return



Solutions to Quizzes

Solution to Quiz: If $y = f(x) = 2x^3 - 15x^2 - 36x + 6$ then

$$\frac{dy}{dx} = 6x^2 - 30x - 36 \quad \text{and} \quad \frac{d^2y}{dx^2} = 12x - 30.$$

There are two stationary points $x = -1$ and $x = 6$ which are obtained from solving the equation

$$0 = \frac{dy}{dx} = 6x^2 - 30x - 36 = 6(x^2 - 5x - 6) = 6(x + 1)(x - 6).$$

At the point $x = -1$ the function value is $y = 25$ and its second derivative is $\frac{d^2y}{dx^2} = 12 \times (-1) - 30 = -42 < 0$. Therefore the point $(-1, 25)$ is a *local maximum*.

For the second stationary point, at $x = 6$, the second derivative is $\frac{d^2y}{dx^2} = 12 \times (6) - 30 = 42 > 0$ and $y = -318$. The point $(6, -318)$ is therefore a *local minimum*.

End Quiz

Solution to Quiz: If $y = x + 1 + \frac{1}{x} = x + 1 + x^{-1}$ then

$$\frac{dy}{dx} = 1 - x^{-2} = 1 - \frac{1}{x^2} \quad \text{and} \quad \frac{d^2y}{dx^2} = -(-2)x^{-3} = \frac{2}{x^3}.$$

The stationary points are found from the equation

$$\frac{dy}{dx} = 1 - \frac{1}{x^2} = \frac{1 - x^2}{x^2} = 0,$$

and this is zero when $1 - x^2 = 0$. There are two solutions in this case, $x = -1$ and $x = 1$.

When $x = -1$ the function value is $y = -1$ and its second derivative is $\frac{d^2y}{dx^2} = \frac{2}{(-1)^3} = -2 < 0$. The point $(-1, -1)$ is a *local maximum*.

When $x = 1$ the function value is $y = 3$ and the second derivative is $\frac{d^2y}{dx^2} = \frac{2}{(1)^3} = 2 > 0$. The point $(1, 3)$ is a *local minimum*.

End Quiz

Solution to Quiz: If $y = \frac{x^2}{2} - \cos(x)$ then

$$\frac{dy}{dx} = x + \sin(x) \quad \text{and} \quad \frac{d^2y}{dx^2} = 1 + \cos(x).$$

The stationary points are found from the equation

$$\frac{dy}{dx} = x + \sin(x) = 0.$$

Since $\sin(0) = 0$, a solution to this equation is $x = 0$. At this point the function value is

$$y = 0 - \cos(0) = 0 - 1 = -1.$$

Its second derivative when $x = 0$ is

$$\frac{d^2y}{dx^2} = 1 + \cos(0) = 1 + 1 = 2 > 0.$$

The point $(0, -1)$ is therefore a *local minimum* of the function

$$y = \frac{x^2}{2} - \cos(x).$$

End Quiz