



Introduction to Partial Differentiation

R Horan & M Lavelle

The aim of this document is to provide a short, self assessment programme for students who wish to acquire a basic understanding of partial differentiation.

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The full range of these packages and some instructions, should they be required, can be obtained from our web page [Mathematics Support Materials](#).

1. Partial Differentiation (Introduction)

In the package on **introductory differentiation**, rates of change of functions were shown to be measured by the *derivative*. Many applications require functions with more than one variable: the ideal gas law, for example, is

$$pV = kT$$

where p is the pressure, V the volume, T the absolute temperature of the gas, and k is a constant. Rearranging this equation as

$$p = \frac{kT}{V}$$

shows that p is a function of T and V . If one of the variables, say T , is kept fixed and V changes, then the derivative of p with respect to V measures the *rate of change* of *pressure* with respect to *volume*. In this case, it is called *the partial derivative of p with respect to V* and written as

$$\frac{\partial p}{\partial V}.$$

Example 1 If $p = \frac{kT}{V}$, find the partial derivatives of p :

- (a) with respect to T , (b) with respect to V .

Solution

(a) This part of the example proceeds as follows:

$$\begin{aligned} p &= \frac{kT}{V}, \\ \therefore \frac{\partial p}{\partial T} &= \frac{k}{V}, \end{aligned}$$

where V is treated as a constant for this calculation.

(b) For this part, T is treated as a constant. Thus

$$\begin{aligned} p &= kT \frac{1}{V} = kTV^{-1}, \\ \therefore \frac{\partial p}{\partial V} &= -kTV^{-2} = -\frac{kT}{V^2}. \end{aligned}$$

The symbol ∂ is used whenever a function with more than one variable is being differentiated but the techniques of *partial* differentiation are exactly the same as for (*ordinary*) differentiation.

Example 2 Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for the function $z = x^2y^3$.

Solution

$$\begin{aligned} z &= x^2y^3 && \text{For the first part } y^3 \text{ is treated as} \\ &&& \text{a constant and the derivative of} \\ \therefore \frac{\partial z}{\partial x} &= 2xy^3, && x^2 \text{ with respect to } x \text{ is } 2x. \\ \text{and } \frac{\partial z}{\partial y} &= x^2 3y^2, && \text{For the second part } x^2 \text{ is treated} \\ &= 3x^2y^2. && \text{as a constant and the derivative} \\ &&& \text{of } y^3 \text{ with respect to } y \text{ is } 3y^2. \end{aligned}$$

EXERCISE 1. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for each of the following functions.

(Click on the green letters for solutions.)

$$\text{(a) } z = x^2y^4, \quad \text{(b) } z = (x^4 + x^2)y^3, \quad \text{(c) } z = y^{\frac{1}{2}} \sin(x).$$

2. The Rules of Partial Differentiation

Since *partial differentiation* is essentially the same as *ordinary differentiation*, the *product*, *quotient* and *chain* rules may be applied.

Example 3 Find $\frac{\partial z}{\partial x}$ for each of the following functions.

$$(a) z = xy \cos(xy), \quad (b) z = \frac{x - y}{x + y}, \quad (c) z = (3x + y)^2.$$

Solution

(a) Here $z = uv$, where $u = xy$ and $v = \cos(xy)$ so the *product rule* applies (see the package on the **Product and Quotient Rules**).

$$\begin{aligned} u &= xy & \text{and} & & v &= \cos(xy) \\ \therefore \frac{\partial u}{\partial x} &= y & \text{and} & & \frac{\partial v}{\partial x} &= -y \sin(xy). \end{aligned}$$

Thus

$$\frac{\partial z}{\partial x} = \frac{\partial u}{\partial x} v + u \frac{\partial v}{\partial x} = y \cos(xy) - xy^2 \sin(xy).$$

(b) Here $z = u/v$, where $u = x - y$ and $v = x + y$ so the *quotient rule* applies (see the package on the **Product and Quotient Rules**).

$$\begin{aligned}u &= x - y & \text{and} & & v &= x + y \\ \therefore \frac{\partial u}{\partial x} &= 1 & \text{and} & & \frac{\partial v}{\partial x} &= 1.\end{aligned}$$

Thus

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{v \frac{\partial u}{\partial x} - u \frac{\partial v}{\partial x}}{v^2} \\ &= \frac{(x + y) - (x - y)}{(x + y)^2} = \frac{2y}{(x + y)^2}.\end{aligned}$$

(c) In this case $z = (3x + y)^2$ so $z = u^2$ where $u = 3x + y$, and the *chain rule* applies (see the package on the **Chain Rule**).

$$\begin{aligned}z &= u^2 & \text{and} & & u &= 3x + y \\ \therefore \frac{\partial z}{\partial u} &= 2u & \text{and} & & \frac{\partial u}{\partial x} &= 3.\end{aligned}$$

Thus

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} = 2(3x + y)3 = 6(3x + y).$$

EXERCISE 2. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for each of the following functions.

(Click on the green letters for solutions.)

(a) $z = (x^2 + 3x) \sin(y)$, (b) $z = \frac{\cos(x)}{y^5}$, (c) $z = \ln(xy)$,

(d) $z = \sin(x) \cos(xy)$, (e) $z = e^{(x^2 + y^2)}$, (f) $z = \sin(x^2 + y)$.

Quiz If $z = \cos(xy)$, which of the following statements is true?

(a) $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y}$,

(b) $\frac{\partial z}{\partial x} = \frac{1}{x} \frac{\partial z}{\partial y}$,

(c) $\frac{1}{y} \frac{\partial z}{\partial x} = \frac{\partial z}{\partial y}$,

(d) $\frac{1}{y} \frac{\partial z}{\partial x} = \frac{1}{x} \frac{\partial z}{\partial y}$.

3. Higher Order Partial Derivatives

Derivatives of order two and higher were introduced in the package on **Maxima and Minima**. Finding higher order derivatives of functions of more than one variable is similar to ordinary differentiation.

Example 4 Find $\frac{\partial^2 z}{\partial x^2}$ if $z = e^{(x^3+y^2)}$.

Solution First differentiate z with respect to x , keeping y constant, then differentiate this function with respect to x , again keeping y constant.

$$\begin{aligned}z &= e^{(x^3+y^2)} \\ \therefore \frac{\partial z}{\partial x} &= 3x^2 e^{(x^3+y^2)} \text{ using the chain rule} \\ \frac{\partial^2 z}{\partial x^2} &= \frac{\partial(3x^2)}{\partial x} e^{(x^3+y^2)} + 3x^2 \frac{\partial(e^{(x^3+y^2)})}{\partial x} \text{ using the product rule} \\ \frac{\partial^2 z}{\partial x^2} &= 6xe^{(x^3+y^2)} + 3x^2(3x^2 e^{(x^3+y^2)}) \\ &= (9x^4 + 6x)e^{(x^3+y^2)}\end{aligned}$$

In addition to both $\frac{\partial^2 z}{\partial x^2}$ and $\frac{\partial^2 z}{\partial y^2}$, when there are two variables there is also the possibility of a *mixed second order derivative*.

Example 5 Find $\frac{\partial^2 z}{\partial x \partial y}$ if $z = e^{(x^3+y^2)}$.

Solution The symbol $\frac{\partial^2 z}{\partial x \partial y}$ is interpreted as $\frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right)$; in words, first differentiate z with respect to y , keeping x constant, then differentiate this function with respect to x , keeping y constant. (It is this differentiation, first with respect to x and then with respect to y , that leads to the name of *mixed derivative*.)

First with x constant $\frac{\partial z}{\partial y} = 2ye^{(x^3+y^2)}$ (using the chain rule.)

Second with y constant $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x} \left(2ye^{(x^3+y^2)} \right)$
 $= 3x^2 2ye^{(x^3+y^2)}$
 $= 6x^2 ye^{(x^3+y^2)}.$

The obvious question now to be answered is: what happens if the order of differentiation is reversed?

Example 6 Find $\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial x} \right)$ if $z = e^{(x^3+y^2)}$.

Solution

First with y constant $\frac{\partial z}{\partial x} = 3x^2 e^{(x^3+y^2)}$ (using the chain rule).

Second with x constant $\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y} \left(3x^2 e^{(x^3+y^2)} \right)$
 $= 2y 3x^2 e^{(x^3+y^2)}$
 $= 6x^2 y e^{(x^3+y^2)} = \frac{\partial^2 z}{\partial x \partial y}.$

As a general rule, when calculating *mixed derivatives* the order of differentiation may be reversed without affecting the final result.

EXERCISE 3. Confirm the statement on the previous page by finding both $\frac{\partial^2 z}{\partial x \partial y}$ and $\frac{\partial^2 z}{\partial y \partial x}$ for each of the following functions, whose first order partial derivatives have already been found in [exercise 2](#). (Click on the [green](#) letters for solutions.)

- (a) $z = (x^2 + 3x) \sin(y)$, (b) $z = \frac{\cos(x)}{y^5}$, (c) $z = \ln(xy)$,
(d) $z = \sin(x) \cos(xy)$, (e) $z = e^{(x^2 + y^2)}$, (f) $z = \sin(x^2 + y)$.

Notation For first and second order partial derivatives there is a compact notation. Thus $\frac{\partial f}{\partial x}$ can be written as f_x and $\frac{\partial f}{\partial y}$ as f_y .

Similarly $\frac{\partial^2 f}{\partial x^2}$ is written f_{xx} while $\frac{\partial^2 f}{\partial x \partial y}$ is written f_{xy} .

Quiz If $z = e^{-y} \sin(x)$, which of the following is $z_{xx} + z_{yy}$?

- (a) $e^{-y} \sin(x)$, (b) 0, (c) $-e^{-y} \sin(x)$, (d) $-e^{-y} \cos(x)$.

4. Quiz on Partial Derivatives

Choose the correct option for each of the following.

Begin Quiz

1. If $z = x^2 + 3xy + y^3$ then $\frac{\partial z}{\partial x}$ is

(a) $2x + 3y + 3y^2$,

(b) $2x + 3x + 3y^2$,

(c) $2x + 3x$,

(d) $2x + 3y$.

2. If $w = 1/r$, where $r^2 = x^2 + y^2 + z^2$, then $xw_x + yw_y + zw_z$ is

(a) $-1/r$,

(b) $1/r$,

(c) $-1/r^2$,

(d) $1/r^2$.

3. If $u = \sqrt{\frac{x}{y}}$ then u_{xx} is

(a) $-\frac{1}{4\sqrt{y^3x^3}}$,

(b) $-\frac{1}{4\sqrt{yx^3}}$,

(c) $-\frac{1}{8\sqrt{y^3x^3}}$,

(d) $-\frac{1}{8\sqrt{yx^3}}$.

End Quiz

Solutions to Exercises

Exercise 1(a) To calculate the partial derivative $\frac{\partial z}{\partial x}$ of the function $z = x^2y^4$, the factor y^4 is treated as a constant:

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x} (x^2y^4) = \frac{\partial}{\partial x} (x^2) \times y^4 = 2x^{(2-1)} \times y^4 = 2xy^4.$$

Similarly, to find the partial derivative $\frac{\partial z}{\partial y}$, the factor x^2 is treated as a constant:

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y} (x^2y^4) = x^2 \times \frac{\partial}{\partial y} (y^4) = x^2 \times 4y^{(4-1)} = 4x^2y^3.$$

Click on the green square to return



Exercise 1(b) To calculate $\frac{\partial z}{\partial x}$ for the function $z = (x^4 + x^2)y^3$, the factor y^3 is treated as a constant:

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x} ((x^4 + x^2)y^3) = \frac{\partial}{\partial x} (x^4 + x^2) \times y^3 = (4x^3 + 2x)y^3.$$

To find the partial derivative $\frac{\partial z}{\partial y}$ the factor $(x^4 + x^2)$ is treated as a constant:

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y} ((x^4 + x^2)y^3) = (x^4 + x^2) \times \frac{\partial}{\partial y} y^3 = 3(x^4 + x^2)y^2.$$

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Exercise 1(c) If $z = y^{\frac{1}{2}} \sin(x)$ then to calculate $\frac{\partial z}{\partial x}$ the $y^{\frac{1}{2}}$ factor is kept constant:

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x} \left(y^{\frac{1}{2}} \sin(x) \right) = y^{\frac{1}{2}} \times \frac{\partial}{\partial x} (\sin(x)) = y^{\frac{1}{2}} \cos(x).$$

Similarly, to evaluate the partial derivative $\frac{\partial z}{\partial y}$ the factor $\sin(x)$ is treated as a constant:

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y} \left(y^{\frac{1}{2}} \sin(x) \right) = \frac{\partial}{\partial y} y^{\frac{1}{2}} \times \sin(x) = \frac{1}{2} y^{-\frac{1}{2}} \sin(x).$$

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Exercise 2(a) The function $z = (x^2 + 3x) \sin(y)$ can be written as $z = uv$, where $u = (x^2 + 3x)$ and $v = \sin(y)$. The partial derivatives of u and v with respect to the variable x are

$$\frac{\partial u}{\partial x} = 2x + 3, \quad \frac{\partial v}{\partial x} = 0,$$

while the partial derivatives with respect to y are

$$\frac{\partial u}{\partial y} = 0, \quad \frac{\partial v}{\partial y} = \cos(y).$$

Applying the *product rule*

$$\frac{\partial z}{\partial x} = \frac{\partial u}{\partial x} v + u \frac{\partial v}{\partial x} = (2x + 3) \sin(y).$$

$$\frac{\partial z}{\partial y} = \frac{\partial u}{\partial y} v + u \frac{\partial v}{\partial y} = (x^2 + 3x) \cos(y).$$

Click on the green square to return



Exercise 2(b)

The function $z = \frac{\cos(x)}{y^5}$ can be written as $z = \cos(x)y^{-5}$.

Treating the factor y^{-5} as a constant and differentiating with respect to x :

$$\frac{\partial z}{\partial x} = -\sin(x)y^{-5} = -\frac{\sin(x)}{y^5}.$$

Treating $\cos(x)$ as a constant and differentiating with respect to y :

$$\frac{\partial v}{\partial y} = \cos(x)(-5y^{-6}) = -5\frac{\cos(x)}{y^6}.$$

Click on the green square to return



Exercise 2(c) The function $z = \ln(xy)$ can be rewritten as (see the package on [logarithms](#))

$$z = \ln(xy) = \ln(x) + \ln(y).$$

Thus the partial derivative of z with respect to x is

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(\ln(x) + \ln(y)) = \frac{\partial}{\partial x} \ln(x) = \frac{1}{x}.$$

Similarly the partial derivative of z with respect to y is

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(\ln(x) + \ln(y)) = \frac{\partial}{\partial y} \ln(y) = \frac{1}{y}.$$

[Click on the green square to return](#)



Exercise 2(d) To calculate the partial derivatives of the function $z = \sin(x) \cos(xy)$ the *product rule* has to be applied

$$\begin{aligned}\frac{\partial z}{\partial x} &= \cos(xy) \frac{\partial}{\partial x} \sin(x) + \sin(x) \frac{\partial}{\partial x} \cos(xy), \\ \frac{\partial z}{\partial y} &= \cos(xy) \frac{\partial}{\partial y} \sin(x) + \sin(x) \frac{\partial}{\partial y} \cos(xy).\end{aligned}$$

Using the *chain rule* with $u = xy$ for the partial derivatives of $\cos(xy)$

$$\begin{aligned}\frac{\partial}{\partial x} \cos(xy) &= \frac{\partial \cos(u)}{\partial u} \frac{\partial u}{\partial x} = -\sin(u)y = -y \sin(xy), \\ \frac{\partial}{\partial y} \cos(xy) &= \frac{\partial \cos(u)}{\partial u} \frac{\partial u}{\partial y} = -\sin(u)x = -x \sin(xy).\end{aligned}$$

Thus the partial derivatives of $z = \sin(x) \cos(xy)$ are

$$\frac{\partial z}{\partial x} = \cos(xy) \cos(x) - y \sin(x) \sin(xy), \quad \frac{\partial z}{\partial y} = -x \sin(x) \sin(xy).$$

Click on the green square to return



Exercise 2(e) To calculate the partial derivatives of $z = e^{(x^2+y^2)}$ the *chain rule* has to be applied with $u = (x^2 + y^2)$:

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{\partial}{\partial u} (e^u) \frac{\partial u}{\partial x} = e^u \frac{\partial u}{\partial x}, \\ \frac{\partial z}{\partial y} &= \frac{\partial}{\partial u} (e^u) \frac{\partial u}{\partial y} = e^u \frac{\partial u}{\partial y}.\end{aligned}$$

The partial derivatives of $u = (x^2 + y^2)$ are

$$\frac{\partial u}{\partial x} = \frac{\partial(x^2)}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = \frac{\partial(y^2)}{\partial y} = 2y.$$

Therefore the partial derivatives of the function $z = e^{(x^2+y^2)}$ are

$$\begin{aligned}\frac{\partial z}{\partial x} &= e^u \frac{\partial u}{\partial x} = 2x e^{(x^2+y^2)}, \\ \frac{\partial z}{\partial y} &= e^u \frac{\partial u}{\partial y} = 2y e^{(x^2+y^2)}.\end{aligned}$$

Click on the green square to return



Exercise 2(f) Applying the *chain rule* with $u = x^2 + y$ the partial derivatives of the function $z = \sin(x^2 + y)$ can be written as

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{\partial}{\partial u} (\sin(u)) \frac{\partial u}{\partial x} = \cos(u) \frac{\partial u}{\partial x}, \\ \frac{\partial z}{\partial y} &= \frac{\partial}{\partial u} (\sin(u)) \frac{\partial u}{\partial y} = \cos(u) \frac{\partial u}{\partial y}.\end{aligned}$$

The partial derivatives of $u = x^2 + y$ are

$$\frac{\partial u}{\partial x} = \frac{\partial x^2}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = \frac{\partial y}{\partial y} = 1.$$

Thus the partial derivatives of the function $z = \sin(x^2 + y)$ are

$$\begin{aligned}\frac{\partial z}{\partial x} &= \cos(u) \frac{\partial u}{\partial x} = 2x \cos(x^2 + y), \\ \frac{\partial z}{\partial y} &= \cos(u) \frac{\partial u}{\partial y} = \cos(x^2 + y).\end{aligned}$$

Click on the green square to return



Exercise 3(a)

From **exercise 2(a)**, the first order partial derivatives of $z = (x^2 + 3x) \sin(y)$ are

$$\frac{\partial z}{\partial x} = (2x + 3) \sin(y), \quad \frac{\partial z}{\partial y} = (x^2 + 3x) \cos(y).$$

The *mixed* second order derivatives are

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial x} ((x^2 + 3x) \cos(y)) = (2x + 3) \cos(y),$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial y} ((2x + 3) \sin(y)) = (2x + 3) \cos(y).$$

Click on the green square to return



Exercise 3(b)

From **exercise 2(b)**, the first order partial derivatives of $z = \frac{\cos(x)}{y^5}$ are

$$\frac{\partial z}{\partial x} = -\frac{\sin(x)}{y^5}, \quad \frac{\partial z}{\partial y} = -5\frac{\cos(x)}{y^6},$$

so the *mixed* second order derivatives are

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial x} \left(-5\frac{\cos(x)}{y^6} \right) = 5\frac{\sin(x)}{y^6},$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial y} \left(-\frac{\sin(x)}{y^5} \right) = 5\frac{\sin(x)}{y^6}.$$

Click on the green square to return



Exercise 3(c)

From **exercise 2(c)**, the first order partial derivatives of $z = \ln(xy)$ are

$$\frac{\partial z}{\partial x} = \frac{1}{x}, \quad \frac{\partial z}{\partial y} = \frac{1}{y}.$$

The *mixed* second order derivatives are

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial x} \left(\frac{1}{y} \right) = 0,$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial y} \left(\frac{1}{x} \right) = 0.$$

Click on the green square to return



Exercise 3(d) From **exercise 2(d)**, the first order partial derivatives of $z = \sin(x) \cos(xy)$ are

$$\frac{\partial z}{\partial x} = \cos(x) \cos(xy) - y \sin(x) \sin(xy), \quad \frac{\partial z}{\partial y} = -x \sin(x) \sin(xy).$$

The *mixed* second order derivatives are

$$\begin{aligned} \frac{\partial^2 z}{\partial x \partial y} &= \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial x} (-x \sin(x) \sin(xy)) \\ &= -\sin(x) \sin(xy) - x \cos(x) \sin(xy) - xy \sin(x) \cos(xy), \\ \frac{\partial^2 z}{\partial y \partial x} &= \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial y} (\cos(x) \cos(xy) - y \sin(x) \sin(xy)) \\ &= -x \cos(x) \sin(xy) - \sin(x) \sin(xy) - xy \sin(x) \cos(xy). \end{aligned}$$

N.B. In the solution above a *product of three functions* has been differentiated. This can be done by using two applications of the *product rule*.

Click on the green square to return



Exercise 3(e) From **exercise 2(e)**, the first order partial derivatives of $z = e^{(x^2+y^2)}$ are

$$\frac{\partial z}{\partial x} = 2xe^{(x^2+y^2)}, \quad \frac{\partial z}{\partial y} = 2ye^{(x^2+y^2)}.$$

The *mixed* second order derivatives are thus

$$\begin{aligned} \frac{\partial^2 z}{\partial x \partial y} &= \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial x} \left(2ye^{(x^2+y^2)} \right) = 4xye^{(x^2+y^2)}, \\ \frac{\partial^2 z}{\partial y \partial x} &= \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial y} \left(2xe^{(x^2+y^2)} \right) = 4yxe^{(x^2+y^2)}. \end{aligned}$$

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Exercise 3(f) From **exercise 2(f)**, the first order partial derivatives of $z = \sin(x^2 + y)$ are

$$\frac{\partial z}{\partial x} = 2x \cos(x^2 + y), \quad \frac{\partial z}{\partial y} = \cos(x^2 + y).$$

The *mixed* second order derivatives are thus

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial x} (\cos(x^2 + y)) = -2x \sin(x^2 + y),$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial y} (2x \cos(x^2 + y)) = -2x \sin(x^2 + y).$$

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Solutions to Quizzes

Solution to Quiz:

To determine which of the options is correct, the partial derivatives of $z = \cos(xy)$ must be calculated. From the calculations of **exercise 2(d)** the partial derivatives of $z = \cos(xy)$ are

$$\frac{\partial}{\partial x} \cos(xy) = \frac{\partial \cos(u)}{\partial u} \frac{\partial u}{\partial x} = -\sin(u)y = -y \sin(xy),$$

$$\frac{\partial}{\partial y} \cos(xy) = \frac{\partial \cos(u)}{\partial u} \frac{\partial u}{\partial y} = -\sin(u)x = -x \sin(xy).$$

Therefore

$$\frac{1}{y} \frac{\partial}{\partial x} \cos(xy) = -\sin(xy) = \frac{1}{x} \frac{\partial}{\partial y} \cos(xy).$$

The other choices, if checked, will be found to be false.

End Quiz

Solution to Quiz:

The first order derivatives of $z = e^{-y} \sin(x)$ are

$$z_x = e^{-y} \cos(x), \quad z_y = -e^{-y} \sin(x),$$

where e^{-y} is kept constant for the first differentiation and $\sin(x)$ for the second. Continuing in this way, the second order derivatives z_{xx} and z_{yy} are given by the expressions

$$\begin{aligned} z_{xx} &= \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial x} (e^{-y} \cos(x)) = -e^{-y} \sin(x), \\ z_{yy} &= \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial y} (-e^{-y} \sin(x)) = e^{-y} \sin(x). \end{aligned}$$

Adding these two equations together gives

$$z_{xx} + z_{yy} = 0.$$

End Quiz