

Powers

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The aim of this document is to provide a short, self assessment programme for students who wish to acquire a basic competence in the use of powers.

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1. Powers (Introduction)

If a is any number and n any *positive* integer (whole number) then the product of a with itself n times, $\underbrace{a \times a \times \cdots \times a}_n$, is called *a raised to the power n* , and written a^n , i.e.,

$$a^n = \underbrace{a \times a \times \cdots \times a}_n.$$

Examples 1

(a) $7^2 = 7 \times 7 = 49$

(b) $2^3 = 2 \times 2 \times 2 = 8$

(c) $3^5 = 3 \times 3 \times 3 \times 3 \times 3 = 243$

The following important rules apply to powers.

$$1. \quad a^m \times a^n = a^{m+n}$$

$$2. \quad a^m \div a^n = a^{m-n}$$

$$3. \quad (a^m)^n = a^{mn}$$

$$4. \quad a^1 = a$$

$$5. \quad a^0 = 1$$

We want these rules to be true for all positive values of a and all values of m and n . We shall first look at the simpler cases.

Examples 2

(a) $10^2 \times 10^3 = (10 \times 10) \times (10 \times 10 \times 10) = 10^5 = 10^{2+3}$.

(b) $2^5 \div 2^3 = 32 \div 8 = 4 = 2^2 = 2^{5-3}$.

(c) $(3^2)^3 = (3 \times 3)^3 = (3 \times 3) \times (3 \times 3) \times (3 \times 3) = 3^6 = 3^{2 \times 3}$

(d) From rule 2 $a^{n+1} \div a^n = a^{(n+1)-n} = a^1$. Also

$$a^{n+1} \div a^n = \frac{\overbrace{a \times a \times \cdots \times a \times a}^{n+1}}{\underbrace{a \times a \times \cdots \times a}_n} = a = a^1.$$

(e) We have $a^n \times a^0 = a^{n+0} = a^n = a^n \times 1$. Thus $a^0 = 1$.

Exercise

Simplify each of the following.

1. $2^3 \times 2^3$

2. $3^{15} \div 3^{12}$

3. $(10^2)^3$

2. Negative Powers

The question now arises as to what we mean by a *negative* power. To interpret this note that

$$a^2 \div a^5 = \frac{a^2}{a^5} = \frac{a \times a}{a \times a \times a \times a \times a} = \frac{1}{a^3}.$$

If rule 2 is to apply, then $a^2 \div a^5 = a^{2-5} = a^{-3}$. Thus $a^{-3} = 1/a^3$. The general rule is

$$a^{-n} = 1/a^n.$$

Examples 3

(a) $10^{-2} = 1/10^2 = 1/100$

(b) $3^{-1} = 1/3^1 = 1/3$

(c) $5^2 \div 5^4 = 5^{(2-4)} = 5^{-2} = 1/5^2 = 1/25$

Exercise

Write each of the following in the form a^k , for some number k .

1. $2^3 \times 2^{-5}$

2. $3^5 \div 3^7$

3. $(10^2)^{-3}$

3. Fractional Powers

If a is a *positive* number, then the *square root* of a is the number which, multiplied by itself, gives a . Thus 3 is the square root of 9 since $3^2 = 9$. We write $3 = \sqrt{9}$. Note that, by definition, $\sqrt{a} \times \sqrt{a} = a$. This gives us a way of interpreting $a^{\frac{1}{2}}$ for, by rule 1,

$$a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{(\frac{1}{2} + \frac{1}{2})} = a^1 = a = \sqrt{a} \times \sqrt{a}$$

so that $a^{\frac{1}{2}} = \sqrt{a}$. The general rule is that, if a is a positive number and n is a positive integer, then

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

where $\sqrt[n]{a}$ is the n -th root of a . We can see this in general for, by rule 3,

$$(a^{\frac{1}{n}})^n = a^{\frac{1}{n} \times n} = a^1 = a.$$

Examples 4

$$(a) \quad 100^{\frac{1}{2}} = \sqrt{100} = 10$$

$$(b) \quad 8^{\frac{1}{3}} = \sqrt[3]{8} = 2$$

$$(c) \quad 27^{\frac{5}{3}} = (27^{\frac{1}{3}})^5 = 3^5 = 243$$

In (c) we have used rule 3, i.e. $a^{\frac{m}{n}} = a^{\frac{1}{n} \times m} = (a^{\frac{1}{n}})^m$, so

$$(a^{\frac{1}{n}})^m = a^{\frac{m}{n}} = (a^m)^{\frac{1}{n}}$$

Quiz. To which of the following does $(8^5)^{\frac{1}{3}}$ simplify?

(a) 8

(b) 16

(c) 24

(d) 32

4. Use of the Rules of Simplification

In this section we shall demonstrate the use of the rules of powers to simplify more complicated expressions.

Examples 5

Simplify each of the following.

$$1. \left[(a^{-3})^{\frac{2}{3}} \right]^{\frac{1}{2}}$$

$$2. \left[\left(x^{-\frac{1}{4}} \right)^8 \right]^{\frac{2}{3}}$$

$$3. \left(x^{\frac{1}{2}} \right)^3 \times \left(x^{-\frac{1}{3}} \right)^2,$$

$$4. \left(\sqrt[4]{x^3} \right)^{\frac{2}{3}} \times \left(\sqrt[5]{x^6} \right)^{\frac{5}{12}}$$

$$5. \left(\frac{a^2}{b^3} \right)^{\frac{1}{3}} \times \left(\frac{b^2}{a^3} \right)^{\frac{1}{2}}$$

5. Quiz on Powers

Begin Quiz

1. $\left(\sqrt[3]{a^5}\right)^{\frac{1}{2}} \times \sqrt[6]{a^{-5}}$

(a) 1

(b) a

(c) $a^{\frac{5}{12}}$

(d) $a^{\frac{5}{6}}$

2. $\left(\frac{a^3}{b^2}\right)^{\frac{1}{2}} \div \left(\frac{b^3}{a^2}\right)^{-\frac{1}{2}}$

(a) $a^{-\frac{1}{2}}b^{-\frac{1}{2}}$

(b) $a^{\frac{1}{2}}b^{-\frac{1}{2}}$

(c) $a^{-\frac{1}{2}}b^{\frac{1}{2}}$

(d) $a^{\frac{1}{2}}b^{\frac{1}{2}}$

3. $\left(\sqrt[4]{b^3}\right)^{\frac{1}{6}} \times \sqrt[9]{b^{-3}} \div \left(\sqrt{b^{-7}}\right)^{\frac{1}{7}}$

(a) $b^{\frac{1}{8}}$

(b) $b^{-\frac{1}{8}}$

(c) $b^{\frac{3}{8}}$

(d) $b^{-\frac{3}{8}}$

End Quiz

Solutions to Quizzes

Solution to Quiz:

Using rule 3, we have

$$(8^5)^{\frac{1}{3}} = 8^{(5 \times \frac{1}{3})} = 8^{(\frac{1}{3} \times 5)} = (8^{\frac{1}{3}})^5 = 2^5 = 32.$$

End Quiz

Solutions to Problems

Problem 1. $2^3 \times 2^3 = 2^{(3+3)} = 2^6 = 64$



Problem 2. $3^{15} \div 3^{12} = 3^{(15-12)} = 3^3 = 27$



Problem 3. $(10^2)^3 = 10^{(2 \times 3)} = 10^6 = 1,000,000$



Problem 1. $2^3 \times 2^{-5} = 2^{(3-5)} = 2^{-2}$, which is $1/4$.



Problem 2. $3^5 \div 3^7 = 3^{(5-7)} = 3^{-2}$, which is $1/9$.



Problem 3. $(10^2)^{-3} = 10^{(2 \times (-3))} = 10^{-6}$, which is $1/1,000,000$.



Problem 1.

Beginning with the innermost bracket, we have, using rule 3,

$$(a^{-3})^{\frac{2}{3}} = a^{-3 \times \frac{2}{3}} = a^{-2}.$$

Then

$$\left[(a^{-3})^{\frac{2}{3}} \right]^{\frac{1}{2}} = [a^{-2}]^{\frac{1}{2}} = a^{-2 \times \frac{1}{2}} = a^{-1}$$



Problem 2.

Beginning again with the innermost bracket, and using rule 3, we have

$$(x^{-\frac{1}{4}})^8 = x^{-\frac{1}{4} \times 8} = x^{-2}.$$

Now if we use rule 3 again we have

$$[x^{-2}]^{\frac{2}{3}} = x^{-2 \times \frac{2}{3}} = x^{-\frac{4}{3}}.$$



Problem 3.

We have

$$\left(x^{\frac{1}{2}}\right)^3 \times \left(x^{-\frac{1}{3}}\right)^2 = x^{\frac{3}{2}} \times x^{-\frac{2}{3}}$$

using rule 3. Now we may use rule 1 and

$$x^{\frac{3}{2}} \times x^{-\frac{2}{3}} = x^{\frac{3}{2} - \frac{2}{3}} = x^{\frac{5}{6}}$$



Problem 4.

Starting with the first term

$$\sqrt[4]{x^3} = (x^3)^{\frac{1}{4}} = x^{\frac{3}{4}}.$$

Thus

$$\left(\sqrt[4]{x^3}\right)^{\frac{2}{3}} = \left(x^{\frac{3}{4}}\right)^{\frac{2}{3}} = x^{\frac{3}{4} \times \frac{2}{3}} = x^{\frac{2}{4}} = x^{\frac{1}{2}}$$

Similarly,

$$\sqrt[5]{x^6} = (x^6)^{\frac{1}{5}} = x^{6 \times \frac{1}{5}} = x^{\frac{6}{5}}$$

so that

$$\left(\sqrt[5]{x^6}\right)^{\frac{5}{12}} = \left(x^{\frac{6}{5}}\right)^{\frac{5}{12}} = x^{\frac{6}{5} \times \frac{5}{12}} = x^{\frac{1}{2}}.$$

Now we have

$$\left(\sqrt[4]{x^3}\right)^{\frac{2}{3}} \times \left(\sqrt[5]{x^6}\right)^{\frac{5}{12}} = x^{\frac{1}{2}} \times x^{\frac{1}{2}} = x^1 = x, .$$



Problem 5.

The first term simplifies as follows.

$$\left(\frac{a^2}{b^3}\right)^{\frac{1}{3}} = \frac{(a^2)^{\frac{1}{3}}}{(b^3)^{\frac{1}{3}}} = \frac{a^{\frac{2}{3}}}{b} = a^{\frac{2}{3}}b^{-1}.$$

Treating the second term,

$$\left(\frac{b^2}{a^3}\right)^{\frac{1}{2}} = \frac{(b^2)^{\frac{1}{2}}}{(a^3)^{\frac{1}{2}}} = \frac{b}{a^{\frac{3}{2}}} = ba^{-\frac{3}{2}}.$$

Thus

$$\left(\frac{a^2}{b^3}\right)^{\frac{1}{3}} \times \left(\frac{b^2}{a^3}\right)^{\frac{1}{2}} = a^{\frac{2}{3}}b^{-1} \times ba^{-\frac{3}{2}} = a^{\frac{2}{3}-\frac{3}{2}} = a^{-\frac{5}{6}}.$$

