Basic Engineering

Binary Numbers 1

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The aim of this document is to provide a short, self assessment programme for students who wish to acquire a basic understanding of the addition and subtraction of binary numbers.

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Last Revision Date: March 17, 2005 Version 1.0

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The full range of these packages and some instructions, should they be required, can be obtained from our web page Mathematics Support Materials.

1. Binary Numbers (Introduction)

The usual arithmetic taught in school uses the *decimal number system*. A number such as 394 (*three hundred and ninety four*) is called a decimal number. This number may be written

$$394 = 3 \times 100 + 9 \times 10 + 4 \times 1 = 3 \times 10^2 + 9 \times 10^1 + 4 \times 10^0 \,.$$

The number is also said to be written *in base 10*. The position of the digits in a particular number indicates the magnitude of the quantity represented and can be assigned a *weight*.

Example 1

In the number 394, the digit 3 has a weight of 100, the digit 9 has a weight of 10 and the digit 4 has a weight of 1.

NB The weight of a number increases from right to left.

Binary numbers are written in base 2 and need only the digits 0,1.

A binary digit (0 or 1) is called a *bit*.

The *weights* of *binary numbers* are in powers of 2 and they also increase from right to left.

Example 2

The binary number 11 is $1 \times 2 + 1 \times 1 = 1 \times 2^1 + 1 \times 2^0$ which in decimal is 3.

NB For the rest of this document a number in decimal form will be written with a subscript 10. Thus 394 will now be written as 394_{10}. The number 11_{10} means the usual decimal number *eleven* whereas the binary number of **example 2** is written 11 or 3_{10}.

Example 3

Convert the binary number 1110101 into a decimal number.

Solution

Binary weight:	2^6	2^5	2^4	2^3	2^2	2^1	2^0
Weight value:	64	32	16	8	4	2	1
Binary digit:	1	1	1	0	1	0	1

The number, in decimal form, is thus

$$1 \times 64 + 1 \times 32 + 1 \times 16 + 0 \times 8 + 1 \times 4 + 0 \times 2 + 1 \times 1 =$$
$$= 64 + 32 + 16 + 4 + 1 = 117_{10}.$$

EXERCISE 1. Convert the following binary numbers into decimal form. (Click on the green letters for the solutions.)

(a) 10,	(b) 101,	(c) 111,	(d) 110,
(e) 1011,	(f) 1111,	(g) 1001,	(h) 1010.

The binary numbers seen so far use only positive powers of 2.
Fractional binary numbers are defined using *negative* powers of 2.

Example 4
Convert the binary number 0.1101 into decimal form.

Solution For this type of binary number the first digit after the decimal point has weight 2^{-1}, the second has weight 2^{-2}, and so on.

Binary weight:	2^{-1}	2^{-2}	2^{-3}	2^{-4}
Weight value:	0.5	0.25	0.125	0.0625
Binary digit	1	1	0	1

The binary number in decimal form is thus

$$1 \times 0.5 + 1 \times 0.25 + 0 \times 0.125 + 1 \times 0.0625$$
$$= 0.5 + 0.25 + 0.0625 = 0.8125_{10}.$$

Exercise 2. Convert each of the following binary numbers into decimal numbers. (Click on the green letters for solutions.)

(a) 0.11, (b) 0.01, (c) 0.101, (d) 0.111,
(e) 1.011, (f) 1.111, (g) 1.001, (h) 10.101.

With non-fractional *two bit numbers* it is possible to count from 0 to 3 inclusively. The numbers are $00 = 0_{10}$, $01 = 1_{10}$, $10 = 2_{10}$ and $11 = 3_{10}$. The *range of numbers counted* is from 0 to 3_{10}.

With non-fractional *three bit numbers* it is possible to count from 0 to 7_{10}. The numbers are $000 = 0_{10}$, $001 = 1_{10}$, $010 = 2_{10}$, $011 = 3_{10}$, $100 = 4_{10}$, $101 = 5_{10}$, $110 = 6_{10}$, $111 = 7_{10}$. The *range of numbers counted* is from 0 to $111 = 7_{10}$.

Quiz What is the *largest number* that can be counted using non-fractional binary numbers with n bits?

(a) 2^{n+1}, (b) 2^{n-1}, (c) $2^n + 1$, (d) $2^n - 1$.

2. Binary Addition

Basic Rules for Binary Addition

0+0	=	0	0 plus 0 equals 0
0+1	=	1	0 plus 1 equals 1
1+0	=	1	1 plus 0 equals 1
1+1	=	10	1 plus 1 equals 0 with a carry of 1 (binary 2)

The technique of addition for binary numbers is similar to that for decimal numbers, except that a 1 is carried to the next column after two 1s are added.

Example 5 Add the numbers 3_{10} and 1_{10} in binary form.

Solution

The numbers, in binary form, are 11 and 01. The procedure is shown on the next page.

$$\begin{array}{r} 11 \\ 01 \\ \hline 100 \end{array}$$

In the right-hand column, $1 + 1 = 0$ with a carry of 1 to the next column.
In the next column, $1 + 0 + 1 = 0$ with a carry of 1 to the next column.
In the left-hand column, $1 + 0 + 0 = 1$.

Thus, in binary, $11 + 01 = 100 = 4_{10}$.

EXERCISE 3. In the questions below, two numbers are given in *decimal* form. In each case, convert both numbers to binary form, add them in binary form and check that the solution is correct by converting the answer to decimal form. (Click on the green letters for solutions.)

(a) 3+3, (b) 7+3, (c) 4+2,
(d) 6+4, (e) 15+12, (f) 28+19,

Quiz What is the result of adding together the **three** binary numbers 101, 110, 1011?

(a) 10110, (b) 11010, (c) 11001, (d) 11110.

3. Binary Subtraction

Basic Rules for Binary Subtraction

$0 - 0$	$=$	0	0 minus 0 equals 0
$1 - 1$	$=$	0	1 minus 1 equals 0
$1 - 0$	$=$	1	1 minus 0 equals 1
$10_2 - 1$	$=$	1	10_2 minus 1 equals 1

Example 6 Subtract $3_{10} = 11$ from $5_{10} = 101$ in binary form.

Solution The subtraction procedure is shown below.

$$\begin{array}{rrr} 1 & 0 & 1 \\ -\ 0 & 1 & 1 \\ \hline & & 0 \end{array} \qquad \begin{array}{rrr} 1 & {}^{1}0 & 1 \\ -\ 0_{1} & 1 & 1 \\ \hline & & 0 \end{array} \qquad \begin{array}{rrr} 1 & {}^{1}0 & 1 \\ -\ 0_{1} & 1 & 1 \\ \hline & 1 & 0 \end{array} \qquad \begin{array}{rrr} 1 & {}^{1}0 & 1 \\ -\ 0_{1} & 1 & 1 \\ \hline 0 & 1 & 0 \end{array}$$

Starting from the left, the first array is the subtraction in the right hand column. In the second array, a 1 is borrowed from the third column for the middle column at the top and paid back at the bottom of the third column. The third array is the subtraction $10 - 1 = 1$ in the middle column. The final array is the subtraction $1 - 1 = 0$ and the final answer is thus $10 = 2_{10}$.

Exercise 4.
In each of the questions below, a subtraction is written in *decimal* form. In each case, convert both numbers to binary form, subtract them in binary form and check that the solution is correct by converting the answer to decimal form. (Click on the green letters for solutions.)

(a) $3-1$, (b) $3-2$, (c) $4-2$,

(d) $6-4$, (e) $9-6$, (f) $9-7$.

Quiz Choose the correct answer from below for the result of the *binary* subtraction $1101-111$.

(a) 110, (b) 101, (c) 111, (d) 11.

4. Quiz on Binary Numbers

Begin Quiz

1. Which of the following is the binary form of 30_{10}?
(a) 10111 (b) 10101, (c) 11011, (d) 11110.

2. Which is the decimal form of the binary number 11.011?
(a) 3.175_{10}, (b) 3.375_{10}, (c) 4.175_{10}, (d) 4.375_{10}.

3. Which of the following is the binary sum $1011 + 1101$?
(a) 11010, (b) 11100, (c) 11000, (d) 10100.

4. Which of the following is the binary subtraction $1101 - 1011$?
(a) 11, (b) 110, (c) 101, (d) 10.

End Quiz

Score: Correct

Solutions to Exercises

Exercise 1(a)

The binary number 10 is

$$10 = 1 \times 2^1 + 0 \times 2^0 = 1 \times 2 + 0 \times 1$$

which in decimal form is 2_{10}.

Click on the green square to return

□

Exercise 1(b)

The binary number 101 is

$$101 = 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 = 1 \times 4 + 1 \times 1$$

which in decimal form is 5_{10}.

Click on the green square to return

□

Exercise 1(c)

The binary number 111 is

$$111 = 1 \times 2^2 + 1 \times 2^1 + 1 \times 2^0 = 1 \times 4 + 1 \times 2 + 1 \times 1$$

which in decimal form is 7_{10}.
Click on the green square to return

□

Exercise 1(d)

The binary number 110 is

$$110 = 1 \times 2^2 + 1 \times 2^1 + 0 \times 2^0 = 1 \times 4 + 1 \times 2$$

which in decimal form is 6_{10}.
Click on the green square to return

□

Exercise 1(e)

The binary number 1011 is

$$1011 = 1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 1 \times 2^0 = 1 \times 8 + 1 \times 2 + 1 \times 1$$

which in decimal form is 11_{10}.

Click on the green square to return

□

Exercise 1(f)

The binary number 1111 is

$$1111 = 1 \times 2^3 + 1 \times 2^2 + 1 \times 2^1 + 1 \times 2^0 = 1 \times 8 + 1 \times 4 + 1 \times 2 + 1 \times 1$$

which in decimal form is 15_{10}.

Click on the green square to return

□

Exercise 1(g)

The binary number 1001 is

$$1001 = 1 \times 2^3 + 0 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 = 1 \times 8 + 1 \times 1$$

which in decimal form is 9_{10}.
Click on the green square to return

□

Exercise 1(h)

The binary number 1010 is

$$1010 = 1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 0 \times 2^0 = 1 \times 8 + 1 \times 2$$

which in decimal form is 10_{10}.
Click on the green square to return

□

Exercise 2(a)

The binary number 0.11 is

$$0.11 = 1 \times 2^{-1} + 1 \times 2^{-2} = 1 \times 0.5 + 1 \times 0.25$$

which in decimal form is 0.75_{10}.
Click on the green square to return

□

Exercise 2(b)

The binary number 0.01 is

$$0.01 = 0 \times 2^{-1} + 1 \times 2^{-2} = 1 \times 0.25 = 0.25\,.$$

Click on the green square to return

□

Exercise 2(c)

The binary number 0.101 is

$$0.101 = 1 \times 2^{-1} + 0 \times 2^{-2} + 1 \times 2^{-3} = 1 \times 0.5 + 1 \times 0.125$$

which in decimal form is 0.625_{10}.

Click on the green square to return

Exercise 2(d)

The binary number 0.111 is

$$0.111 = 1 \times 2^{-1} + 1 \times 2^{-2} + 1 \times 2^{-3} = 1 \times 0.5 + 1 \times 0.25 + 1 \times 0.125$$

which in decimal form is 0.875_{10}.

Click on the green square to return

□

Exercise 2(e)

The binary number 1.011 is

$$\begin{aligned} 1.011 &= 1\times 2^0 + 0\times 2^{-1} + 1\times 2^{-2} + 1\times 2^{-3} \\ &= 1\times 1 + 1\times 0.25 + 1\times 0.125 \end{aligned}$$

which in decimal form is 1.375_{10}.

Click on the green square to return

Exercise 2(f)

The binary number 1.111 is

$$\begin{aligned} 1.111 &= 1 \times 2^0 + 1 \times 2^{-1} + 1 \times 2^{-2} + 1 \times 2^{-3} \\ &= 1 \times 1 + 1 \times 0.5 + 1 \times 0.25 + 1 \times 0.125 \end{aligned}$$

which in decimal form is 1.875_{10}.

Click on the green square to return

Exercise 2(g)

The binary number 1.001 is

$$\begin{aligned} 1.001 &= 1 \times 2^0 + 0 \times 2^{-1} + 0 \times 2^{-2} + 1 \times 2^{-3} \\ &= 1 \times 1 + 1 \times 0.125 \end{aligned}$$

which in decimal form is 1.125_{10}.

Click on the green square to return

□

Exercise 2(h)

The binary number 10.101 is

$$\begin{aligned} 10.101 &= 1 \times 2^1 + 0 \times 2^0 + 1 \times 2^{-1} + 0 \times 2^{-2} + 1 \times 2^{-3} \\ &= 2 + 1 \times 0.5 + 1 \times 0.125 \end{aligned}$$

which in decimal form is 2.625_{10}.

Click on the green square to return

□

Exercise 3(a)

To add the numbers $3 + 3$ in binary form first convert the number 3_{10} to binary form. The result is $3_{10} = 11$. The sum is shown below.

$$\begin{array}{r} 11 \\ 11 \\ \hline 110 \end{array}$$

In the right-hand column, $1 + 1 = 0$ with a carry of 1 to the next column.
In the next column, $1 + 1 + 1 = 0 + 1 = 1$ with a carry of 1 to the next column.
In the left-hand column, $1 + 0 + 0 = 1$.

Thus, in binary, $11 + 11 = 110$. In decimal form this is 6_{10}.

Click on the green square to return

Exercise 3(b)

To add the numbers $7 + 3$ in binary form, note that the binary form of 3_{10} is $3_{10} = 11$, while $7_{10} = 111$. The sum $7 + 3$ in binary form is shown below.

$$\begin{array}{r} 11 \\ 111 \\ \hline 1010 \end{array}$$

In the right-hand column, $1 + 1 = 0$ with a carry of 1 to the next column.
In the next column, $1 + 1 + 1 = 0 + 1 = 1$ with a carry of 1 to the next column.
In the left-hand column, $1 + 1 + 0 = 0$ with a carry of 1 to the next column.

Thus, in binary, $11 + 111 = 1010$. In decimal form this is 10_{10}.

Click on the green square to return

Exercise 3(c)

To add the numbers $4 + 2$ in binary form, note that $4_{10} = 100$, while $2_{10} = 10$. The sum $4 + 2$, in binary form is shown below.

$$\begin{array}{r} 100 \\ 10 \\ \hline 110 \end{array}$$

In the right-hand column, $0 + 0 = 0$.
In the next column, $0 + 1 = 1$.
In the left-hand column, $1 + 0 = 1$.

Thus, in binary, $100 + 10 = 110$, which in decimal form is 6_{10}.

Click on the green square to return

Exercise 3(d)

To add the numbers $6 + 4$ in binary form first convert the numbers to binary form. They are $6_{10} = 110$ and $4_{10} = 100$. The sum $6 + 4$ in binary form is shown below.

$$\begin{array}{r} 110 \\ 100 \\ \hline 1010 \end{array}$$

In the right-hand column, $0 + 0 = 0$.
In the next column, $1 + 0 = 1$.
In the left-hand column, $1+1 = 0$ with a carry of 1 to the next column.

Thus, in binary, $110 + 100 = 1010$. In decimal form this is 10_{10}.

Click on the green square to return

Exercise 3(e)

To add the numbers $15 + 12$, in binary form, note that

$$15_{10} = 8 + 4 + 2 + 1 = 1 \times 2^3 + 1 \times 2^2 + 1 \times 2^1 + 1 \times 2^0 = 1111\,,$$

while

$$12_{10} = 8 + 4 = 1 \times 2^3 + 1 \times 2^2 = 1100\,.$$

The sum $15 + 12$ in binary form is shown below.

$$\begin{array}{r} 1111 \\ 1100 \\ \hline 11011 \end{array}$$

Note that in the third column, $1 + 1 = 0$ with a carry of 1 to the next column. In the left-hand column, $1 + 1 + 1 = 1$ with a of carry 1 to the next column.

Thus, in binary, $1111 + 1100 = 11011$, which in decimal form is $11011 = 2^4 + 2^3 + 2^1 + 2^0 = 16 + 8 + 2 + 1 = 27_{10}$.

Click on the green square to return

Exercise 3(f)

To add the numbers $28 + 19$ in binary form convert them both to binary form.

$$28_{10} = 16 + 8 + 4 = 1 \times 2^4 + 1 \times 2^3 + 1 \times 2^2 = 11100\,,$$

while

$$19_{10} = 16 + 2 + 1 = 1 \times 2^4 + 1 \times 2^1 + 1 \times 2^0 = 10011\,.$$

The sum $28 + 19$ in binary form is shown below.

$$\begin{array}{r} 11100 \\ 10011 \\ \hline 101111 \end{array}$$

Note that in the left-hand column, $1 + 1 = 0$ with a carry of 1 to the next column.

Thus, in binary, $11100 + 10011 = 101111$, which in decimal form is $101111 = 2^5 + 2^3 + 2^2 + 2^1 + 2^0 = 32 + 8 + 4 + 2 + 1 = 47_{10}$.

Click on the green square to return

Exercise 4(a)

To find $3 - 1$ in binary form, recall that $3_{10} = 11$, while $1_{10} = 1$. The subtraction, in binary form, is shown below.

$$\begin{array}{r} 11 \\ -\ \ 1 \\ \hline 10 \end{array}$$

In the right-hand column, $1 - 1 = 0$.
In the next column, $1 - 0 = 1$.

Thus $11 - 1 = 10$ which, in decimal form, is 2_{10}.

Click on the green square to return

Exercise 4(b)

To find the difference $3 - 2$ in binary form, convert the numbers into binary form, i.e. $3_{10} = 11$ and $2_{10} = 10$. The subtraction, in binary form, is shown below.

$$\begin{array}{r} 11 \\ -\ 10 \\ \hline 01 \end{array}$$

In the right-hand column $1 - 0 = 1$.
In the next column $1 - 1 = 0$.

Thus, in binary form, $11 - 10 = 1$. In decimal form this is 1_{10}.

Click on the green square to return

Exercise 4(c)

To find the difference $4 - 2$ in binary form, first convert the numbers into binary form. Thus $4_{10} = 100$ and $2_{10} = 10$. The subtraction, in binary form, is shown below.

$$\begin{array}{r} 100 \\ -\ 010 \\ \hline 10 \end{array}$$

In the right-hand column, $0 - 0 = 0$.
In the next column, a 1 is borrowed from the third column so $10 - 1 = 1$.
In the left-hand column, taking into account the paid back 1, we have $1 - (1 + 0) = 0$.

Thus, in binary, $100 - 10 = 10$. In decimal form this is 2_{10}.

Click on the green square to return

Exercise 4(d)

To find the difference $6 - 4$ in binary form, note that $6_{10} = 110$ and $4_{10} = 100$. The subtraction, in binary form, is shown below.

$$\begin{array}{r} 110 \\ -\ 100 \\ \hline 010 \end{array}$$

In the right-hand column $0 - 0 = 0$.
In the next column $1 - 0 = 1$.
In the left-hand column $1 - 1 = 0$.

Thus, in binary, $110 - 100 = 10$. In decimal form this is 2_{10}.

Click on the green square to return

Exercise 4(e)

To find the difference $9-6$ in binary form note that $9_{10} = 1001$ and $6_{10} = 110$. The subtraction, in binary form, is shown below.

$$\begin{array}{r} 1001 \\ -\ 110 \\ \hline 011 \end{array}$$

In the right-hand column $1-0=1$.
In the next column, borrow a 1 from the third column (at the top) and pay it back at the bottom of the third column. Then $10-1=1$. The bottom of the third column is now $1+1=10$. The final step is thus $10-10=00$.

Thus, in binary, $1001-110=11$. In decimal form this is 3_{10}.

Click on the green square to return

□

Exercise 4(f)

To find the difference $9 - 7$ in binary form note that $9_{10} = 1001$ and $7_{10} = 111$. The subtraction, in binary form, is shown below.

$$\begin{array}{r} 1001 \\ -\ 111 \\ \hline 010 \end{array}$$

In the right-hand column, $1 - 1 = 0$.
In the second column borrow a 1 from (the top of) the third column and pay it back at the bottom of the third column. The second column is now $10 - 1 = 1$.
The bottom of the third column now becomes $1 + 1 = 10$.
The final subtraction is now $10 - 10 = 00$.

Thus, in binary, $1001 - 110 = 10$. In decimal form this is 2_{10}.

Click on the green square to return

Solutions to Quizzes

Solution to Quiz: The maximal decimal number N_n that can be represented by the non-fractional binary number with n bits using only the digit 1 in each of n positions, is written $N_n = \underbrace{11\cdots11}_{n}$.

In the **introduction** it was shown that $N_1 = 1$, $N_2 = 3$, and $N_3 = 7$. By direct calculation it can be checked that these numbers can be obtained from the formula $N_n = 2^n - 1$ for $n = 1, 2, 3$ respectively. This can be checked for other values of n.

For those interested, the proof of the general rule is as shown below.

$$N_n \quad = \quad \underbrace{11\cdots11}_{n} = 2^{n-1} + \cdots 2 + 1 \ .$$

This is a geometric progression with common ratio 2 and its sum is $N_n = (2^{(n-1)+1} - 1)/(2-1) = 2^n - 1$. End Quiz

Solution to Quiz:
The addition of the three binary numbers 101, 110, 1011 is shown below.

$$\begin{array}{r} 101 \\ 110 \\ 1011 \\ \hline 10110 \end{array}$$

Note that in performing the summation, we use $1 + 1 = 0$ with a carry of 1 to the next column.

The result is $10110 = 2^4 + 2^2 + 2^1 = 16 + 4 + 2 = 22$. Converting each number to decimal form $101 = 5_{10}$, $110 = 6_{10}$ and $1011 = 11_{10}$ which can be used to verify the result.

End Quiz

Solution to Quiz:
The subtraction $1101 - 111$ is given below.

$$\begin{array}{r} 1101 \\ -\ 111 \\ \hline 110 \end{array}$$

In the right-hand column, $1 - 1 = 0$.
In the second column a 1 is borrowed from the third column (at the top) and paid back at the bottom of the third column, resulting in $10 - 1 = 1$.
The bottom of the third column is now $1+1 = 10$. This leaves the subtraction $11 - 10 = 1$.

In decimal form the result of the subtraction is $110 = 2^2 + 2 = 6_{10}$. Converting the numbers to decimal form, $1101 = 2^3 + 2^2 + 1 = 13_{10}$ and $111 = 2^2 + 2 + 1 = 7_{10}$, confirming this result.

End Quiz