

2 - D STEADY STATE HEAT CONDUCTION

In 1-D heat conduction the controlling equation was found to be:-

$$\frac{\partial^2 T}{\partial x^2} = 0$$

In an isotropic material the same is also true in the direction at right angles to the x-direction, so we can deduce that the controlling equation for 2-D conduction is:-

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

We can obviously extend this to 3-D, and also include two further possibilities:-

- (a) thermal storage within an element; and
- (b) heat generation within an element.

This would give us the general equation for non-steady state heat conduction in an isotropic material:-

$$\frac{\partial T}{\partial t} = \frac{1}{r C_p} \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + \frac{\dot{Q}'''}{r C_p}$$

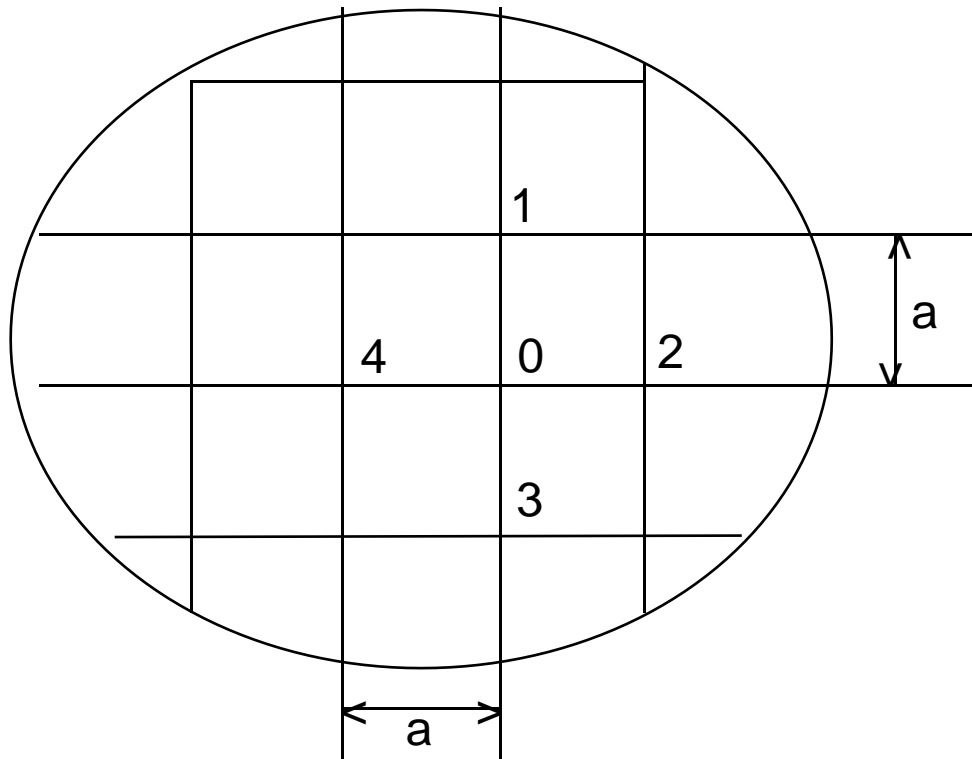
$\frac{1}{r C_p}$ is the THERMAL DIFFUSIVITY of the material (a)

$$\frac{\partial T}{\partial t} = a D^2 T + \frac{\dot{Q}'''}{r C_p}$$

Exact solutions to this equation are only possible in a limited number of well defined cases. Otherwise we must resort to numerical (computer) solutions.

The equation for 2-D steady conditions can be solved using finite difference techniques.

Consider the area to be divided up by a rectangular grid where the grid lines are spaced 'a' apart:-



The temperature gradient between any two nodes is given by:-

$$\frac{T_0 - T_4}{a} \text{ between } 4 \text{ \& } 0 ; \quad \frac{T_2 - T_0}{a} \text{ between } 0 \text{ \& } 2$$

Therefore at 0:-

$$\frac{\partial^2 T}{\partial x^2} = \frac{\frac{T_2 - T_0}{a} - \frac{T_0 - T_4}{a}}{a}$$

$$\text{or } \frac{\partial^2 T}{\partial x^2} = \frac{T_2 + T_4 - 2T_0}{a^2}$$

$$\text{also } \frac{\partial^2 T}{\partial y^2} = \frac{T_1 + T_3 - 2T_0}{a^2}$$

therefore $\frac{\nabla^2 T}{\nabla x^2} + \frac{\nabla^2 T}{\nabla y^2} = \frac{T_1 + T_2 + T_3 + T_4 - 4T_o}{a^2} = 0$

or

$$T_1 + T_2 + T_3 + T_4 - 4T_o = 0$$

or

$$T_o = (T_1 + T_2 + T_3 + T_4) / 4$$

This condition must be satisfied at every internal point.

If heat was being generated within the element by electrical heating then $\dot{Q}''' = i^2 r$ where:-

i = current density (amps/m²); and

r = electrical resistivity (Wm)

then $T_1 + T_2 + T_3 + T_4 - 4T_o = - \frac{a^2 \dot{Q}'''}{k}$

These equations may be solved by spreadsheet iteration - 'updating' each temperature 'cell' in sequence .

Alternatively an array of simultaneous equations may be set up and solved by Gaussian elimination, either by hand (!) or by computer.

BOUNDARY CONDITIONS

The above equations apply to interior points. Special conditions exist on boundaries which must be dealt with .

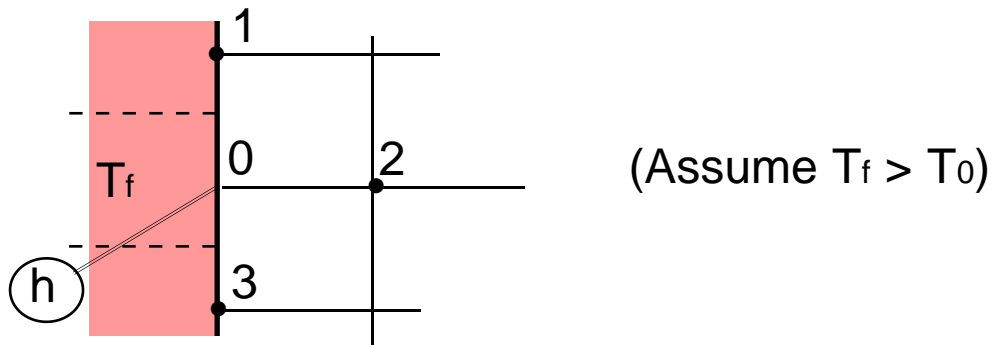
Common boundary conditions are :-

- (a) constant surface temperature (isothermal) ; or
- (b) convective heat transfer.

On an isothermal boundary each node has the same temperature ie $T = \text{constant}$.

The solution can then proceed using the above equation for the internal points.

On a convective boundary, let the fluid temperature be T_f and the local convective surface heat transfer coefficient be



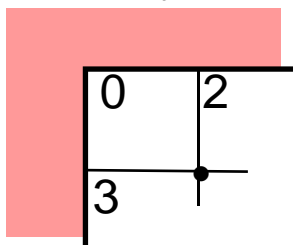
At 0 : Heat transfer 'in' = Heat transfer 'out'

$$ha(T_f - T_0) = l a \left(\frac{T_0 - T_2}{a} \right) + \frac{1}{2} l a \left(\frac{T_0 - T_1}{a} \right) + \frac{1}{2} l a \left(\frac{T_0 - T_3}{a} \right)$$

$$\text{or } \boxed{\frac{1}{2}T_1 + \frac{1}{2}T_3 + T_2 + \frac{ha}{l} T_f - (2 + \frac{ha}{l}) T_0 = 0}$$

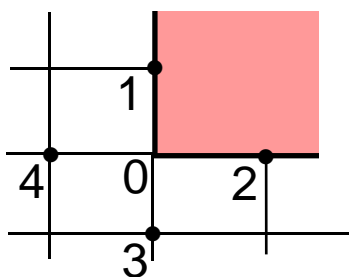
$$\text{or } T_0 = \frac{1}{2 + \frac{ha}{l}} \left(\frac{1}{2} T_1 + \frac{1}{2} T_3 + T_2 + \frac{ha}{l} T_f \right)$$

Similarly it can be shown that for :-



$$\frac{1}{2}T_2 + \frac{1}{2}T_3 + \frac{ha}{l} T_f - (1 + \frac{ha}{l}) T_0 = 0$$

$$\text{or } T_0 = \frac{1}{1 + \frac{ha}{l}} \left(\frac{1}{2} T_2 + \frac{1}{2} T_3 + \frac{ha}{l} T_f \right)$$



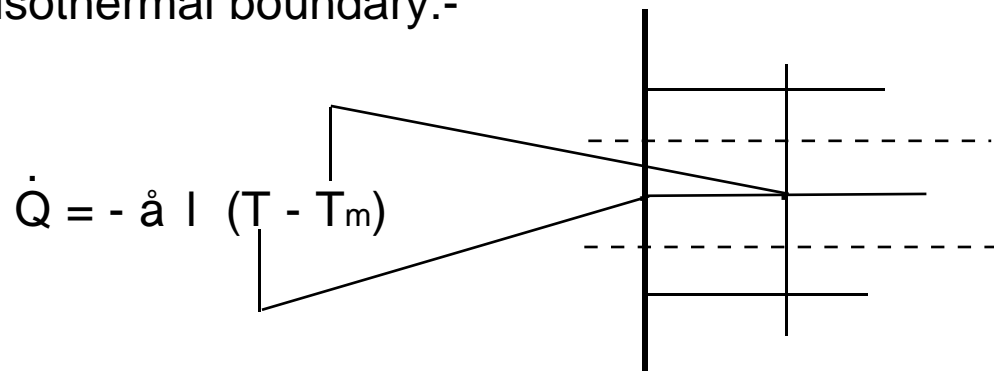
$$\frac{1}{2}T_1 + \frac{1}{2}T_2 + T_3 + T_4 + \frac{ha}{l} T_f - (3 + \frac{ha}{l}) T_0 = 0$$

$$\text{or } T_0 = \frac{1}{3 + \frac{ha}{l}} \left(\frac{1}{2} T_1 + \frac{1}{2} T_2 + T_3 + T_4 + \frac{ha}{l} T_f \right)$$

HEAT TRANSFER RATE

Solutions to the above equations give the temperature distribution within the material. In order to find the heat transfer rates we must sum all of the rates into (or out of) the element.

For an isothermal boundary:-



For a convective boundary:-

