Rules of Mixture for Elastic Properties

'Rules of Mixtures' are mathematical expressions which give some property of the composite in terms of the properties, quantity and arrangement of its constituents.

They may be based on a number of simplifying assumptions, and their use in design should tempered with extreme caution!

Density

For a general composite, total volume V, containing masses of constituents M_a, M_b, M_c,... the composite density is

$$\rho = \frac{M_a + M_b + M_c + \ldots}{V} = \frac{M_a}{V} + \frac{M_b}{V} + \ldots$$

In terms of the densities and volumes of the constituents:

$$\rho = \frac{V_a \rho_a}{V} + \frac{V_b \rho_b}{V} + \frac{V_c \rho_c}{V} + \ldots$$

Density

But $v_a / V = V_a$ is the volume fraction of the constituent a, hence:

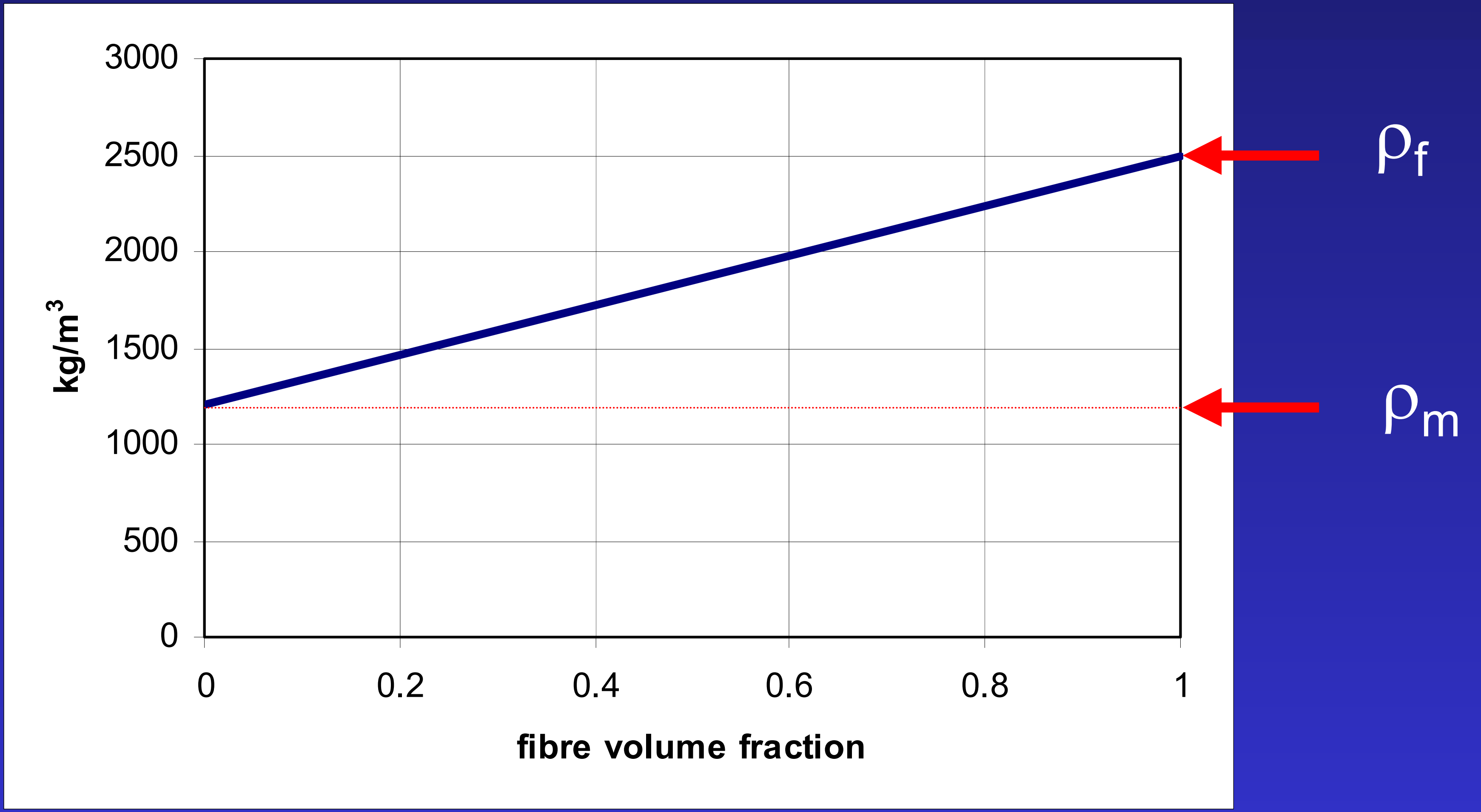
$$\rho = V_a \rho_a + V_b \rho_b + V_c \rho_c + \ldots$$

For the special case of a fibre-reinforced matrix:

$$\rho = V_f \rho_f + V_m \rho_m = V_f \rho_f + (1 - V_f) \rho_m = V_f (\rho_f - \rho_m) + \rho_m$$

since $V_f + V_m = 1$

Rule of mixtures density for glass/epoxy composites

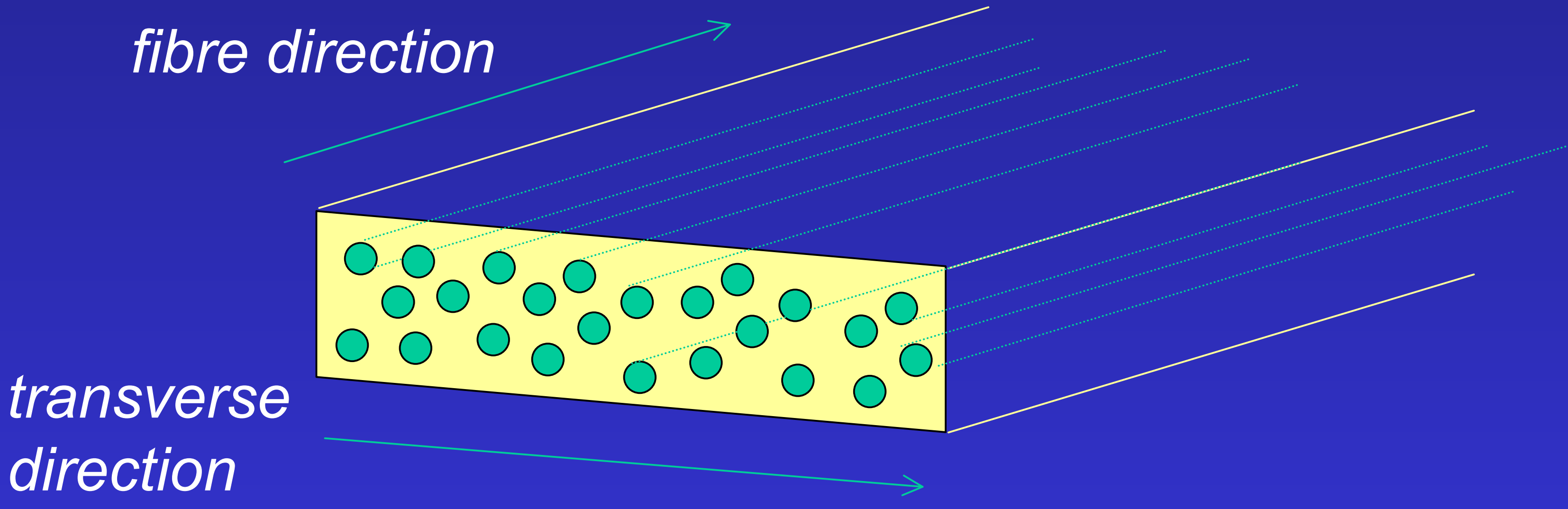


Micromechanical models for stiffness

Unidirectional ply

Unidirectional fibres are the simplest arrangement of fibres to analyse.

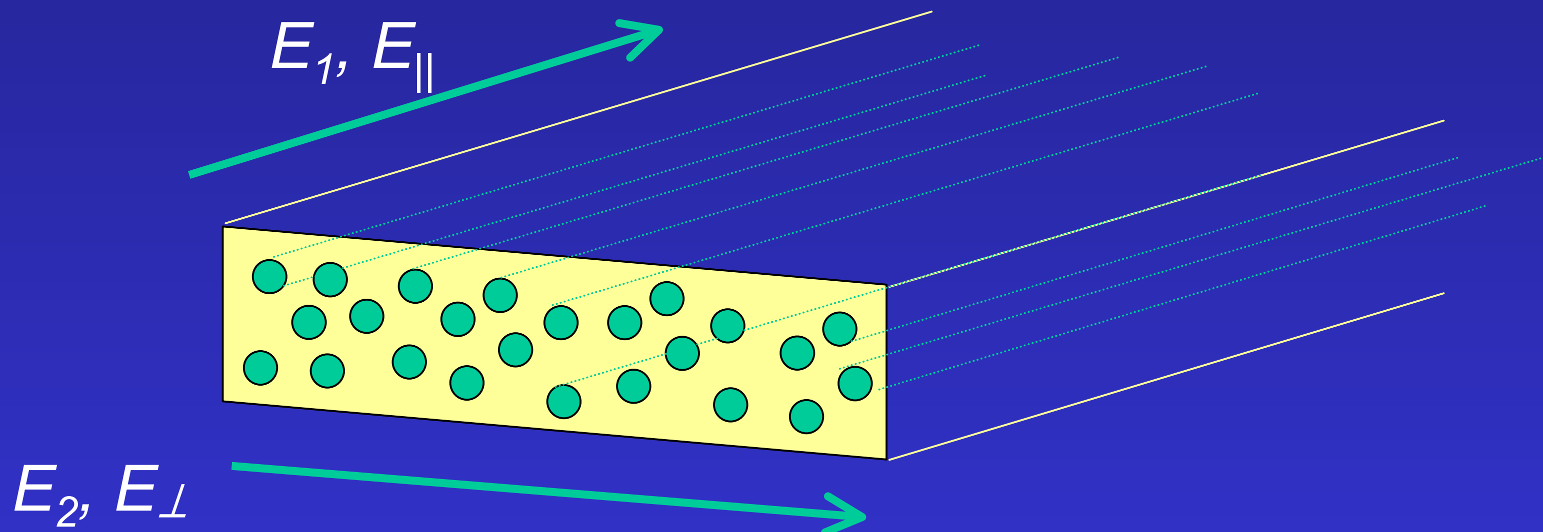
They provide maximum properties in the fibre direction, but minimum properties in the transverse direction.



Unidirectional ply

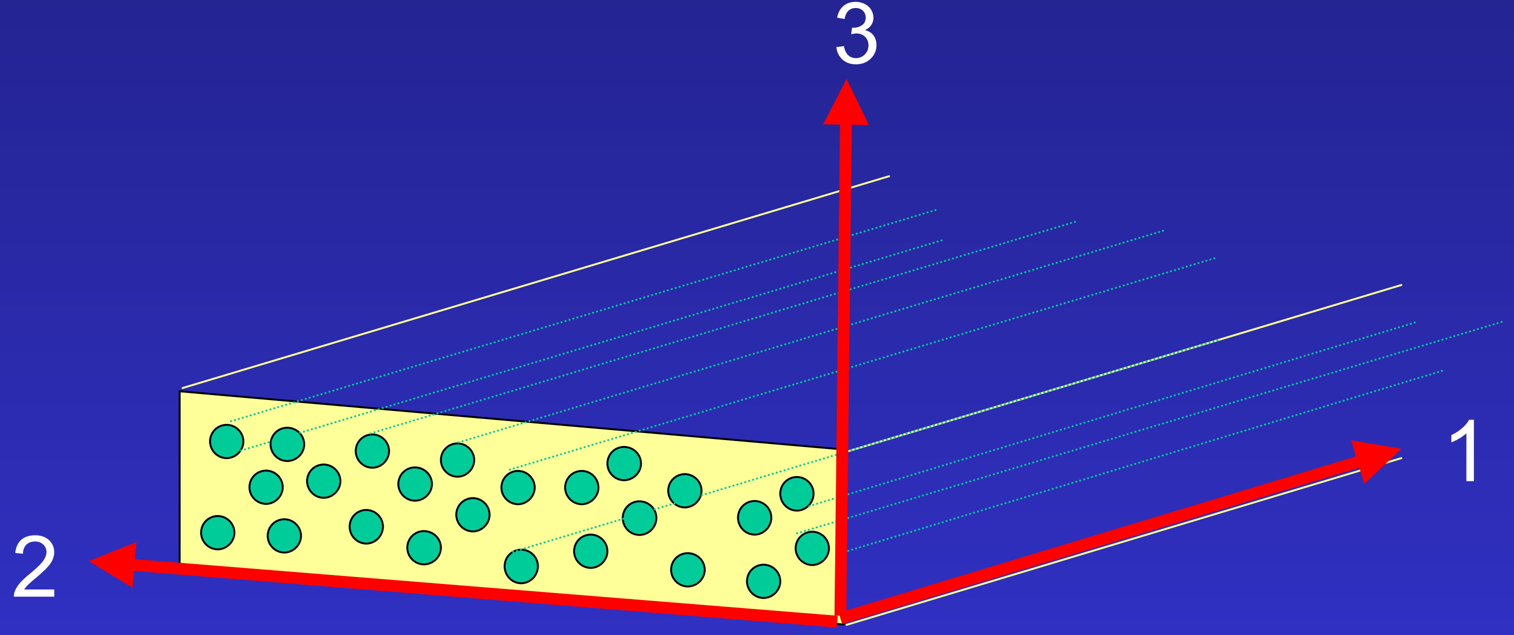
We expect the unidirectional composite to have different tensile moduli in different directions.

These properties may be labelled in several different ways:



Unidirectional ply

By convention, the principal axes of the ply are labelled ‘1, 2, 3’. This is used to denote the fact that ply may be aligned differently from the cartesian axes x, y, z.



Unidirectional ply - longitudinal tensile modulus

We make the following assumptions in developing a rule of mixtures:

- Fibres are uniform, parallel and continuous.
- Perfect bonding between fibre and matrix.
- Longitudinal load produces equal strain in fibre and matrix.

Unidirectional ply - longitudinal tensile modulus

- A load applied in the fibre direction is shared between fibre and matrix:

$$F_1 = F_f + F_m$$

- The stresses depend on the cross-sectional areas of fibre and matrix:

$$\sigma_1 A = \sigma_f A_f + \sigma_m A_m$$

where A $(= A_f + A_m)$ is the total cross-sectional area of the ply

Unidirectional ply - longitudinal tensile modulus

- Applying Hooke's law:
  $E_1 \varepsilon_1 A = E_f \varepsilon_f A_f + E_m \varepsilon_m A_m$
  where Poisson contraction has been ignored

- But the strain in fibre, matrix and composite are the same, so
  $\varepsilon_1 = \varepsilon_f = \varepsilon_m$, and:

  $E_1 A = E_f A_f + E_m A_m$

Unidirectional ply - longitudinal tensile modulus

Dividing through by area A:

$$E_1 = E_f\,(A_f / A) + E_m\,(A_m / A)$$

But for the unidirectional ply, (A_f / A) and (A_m / A) are the same as volume fractions V_f and $V_m = 1-V_f$. Hence:

$$\boldsymbol{E_1 = E_f V_f + E_m (1-V_f)}$$

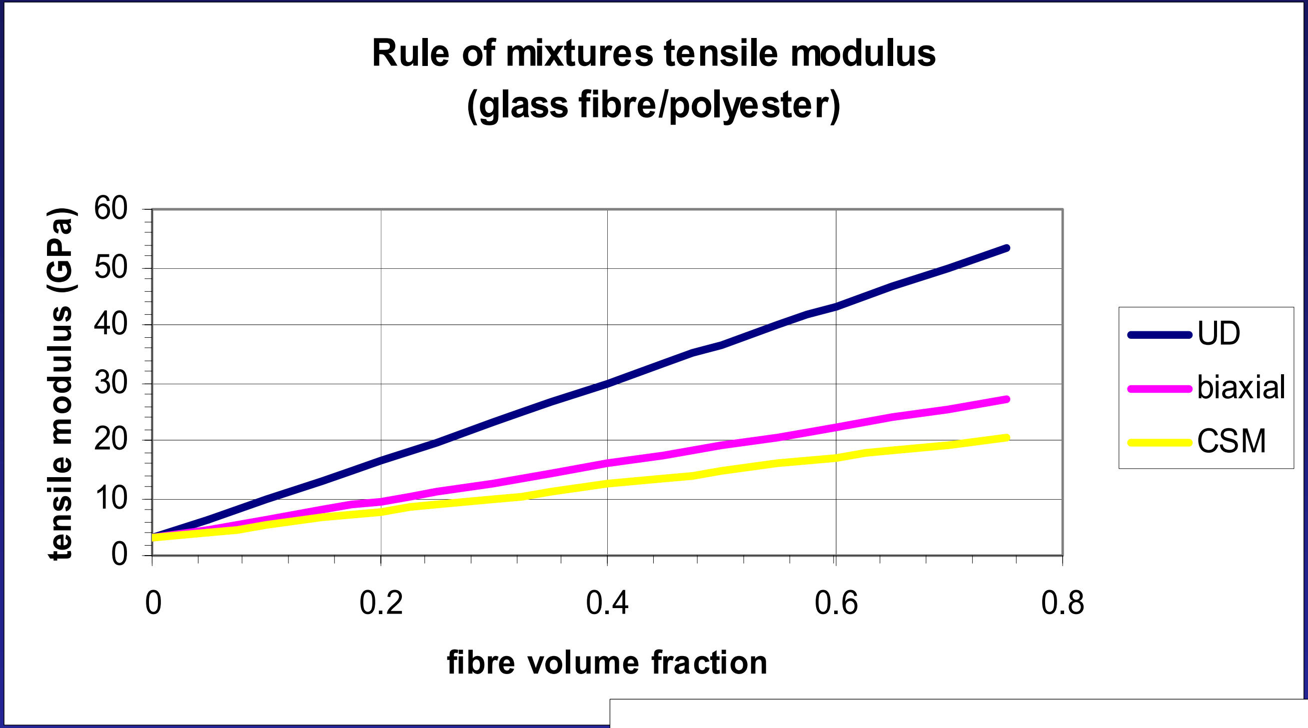
Unidirectional ply - longitudinal tensile modulus

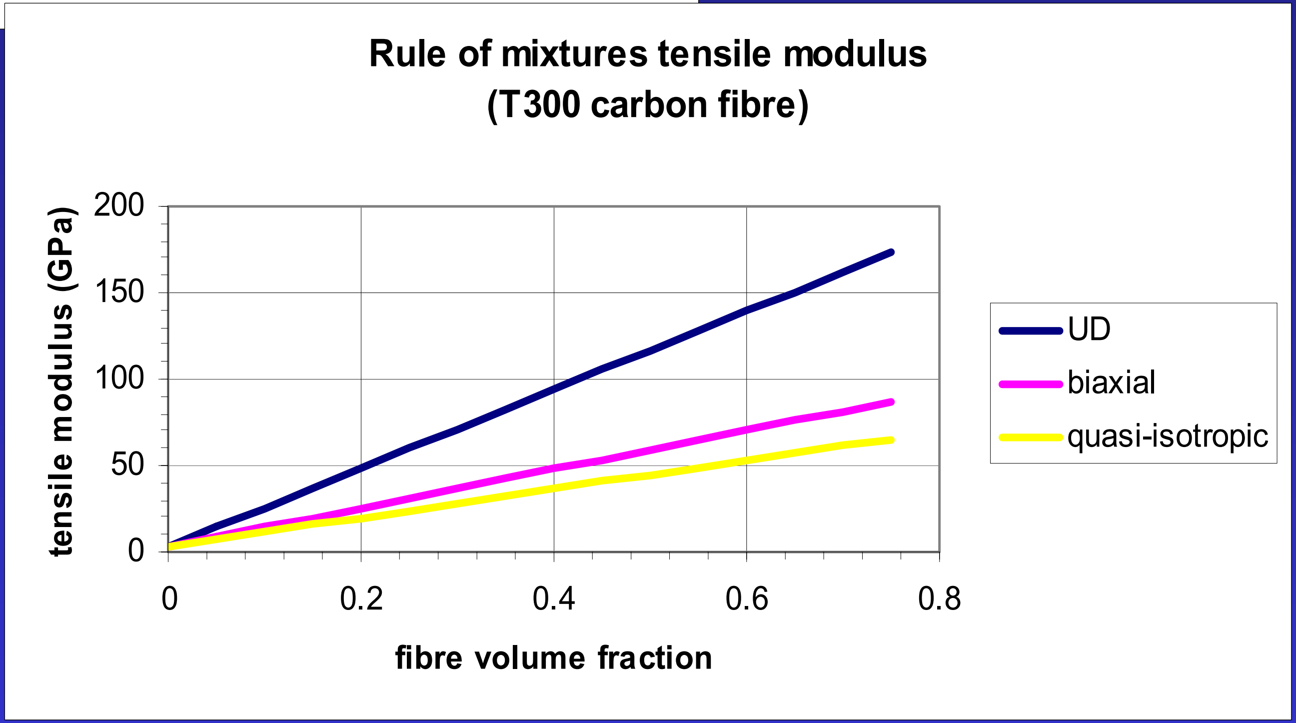
$$E_1 = E_f V_f + E_m (1 - V_f)$$

Note the similarity to the rules of mixture expression for density.

In polymer composites, $E_f >> E_m$, so

$$E_1 \approx E_f V_f$$

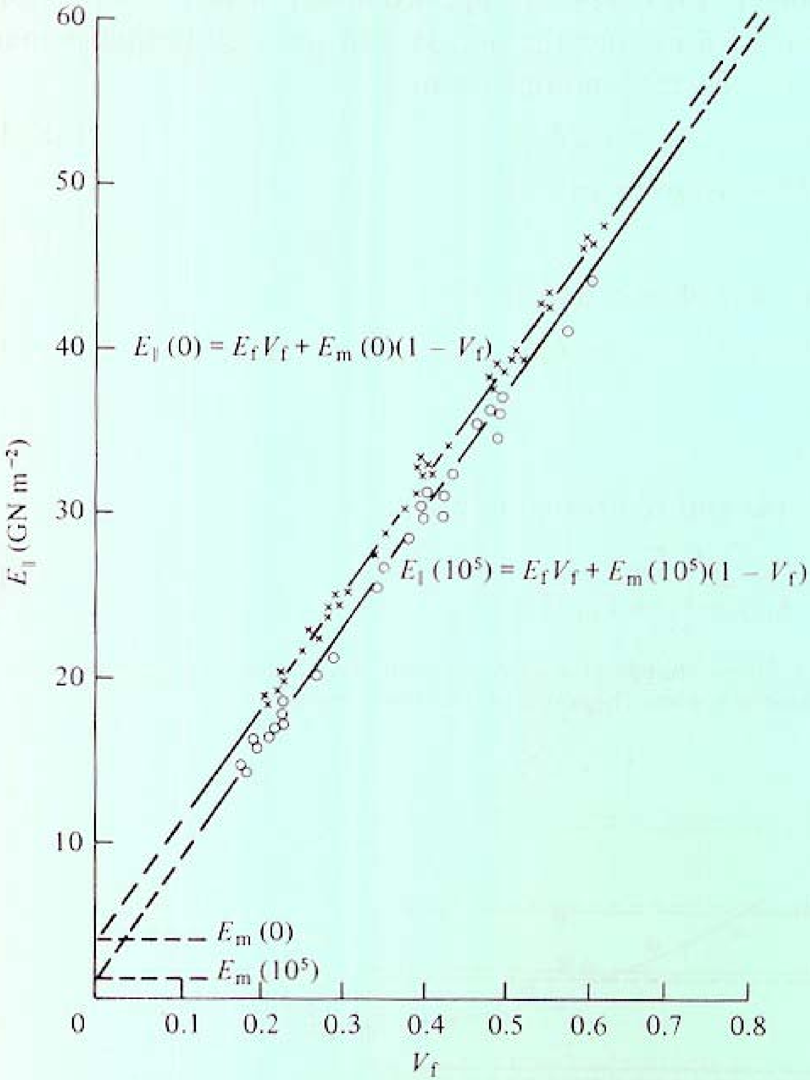

Rule of mixtures tensile modulus
(glass fibre/polyester)
tensile modulus (GPa)
60
50
40
30
20
10
0
0
0.2
0.4
0.6
0.8
fibre volume fraction
UD
biaxial
CSM


Rule of mixtures tensile modulus
(T300 carbon fibre)
tensile modulus (GPa)
200
150
100
50
0
0
0.2
0.4
0.6
0.8
fibre volume fraction
UD
biaxial
quasi-isotropic

This rule of mixtures is a good fit to experimental data

(source: Hull, Introduction to Composite Materials, CUP)

Fig. 5.2. Elastic moduli measured parallel to fibres of undirectional laminae of glass fibres and polyester resin with different V_f. (From Brintrup Dr-Ing thesis 1975, Technischen Hochschule, Aachen.)

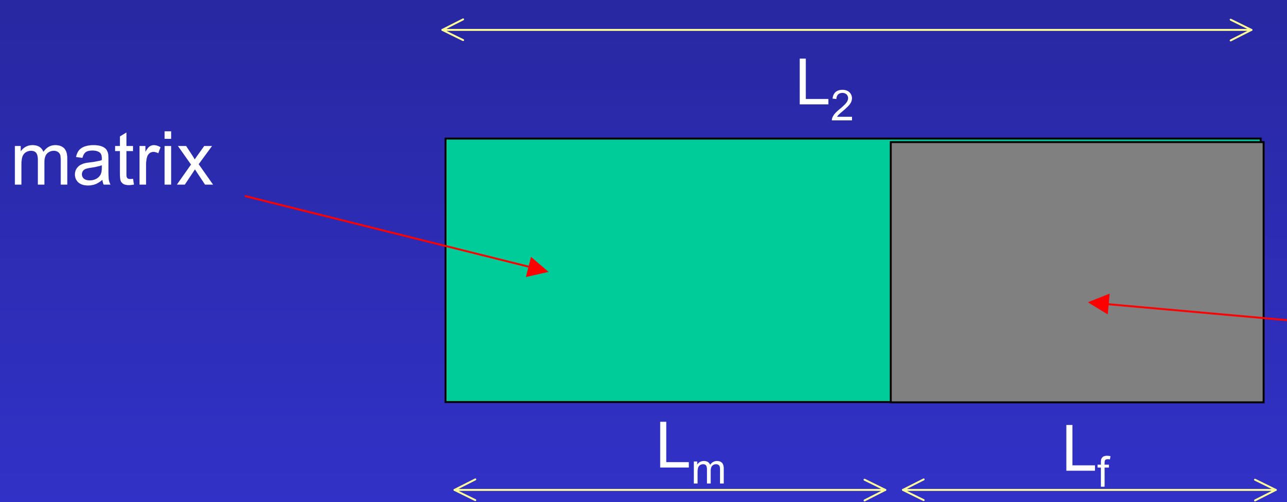


† Notation – for a unidirectional lamina $E_1 = E_\parallel$, $E_2 = E_\perp$, $G_{12} = G_{\parallel\perp}$ and $\nu_{12} = \nu_{\parallel\perp}$.

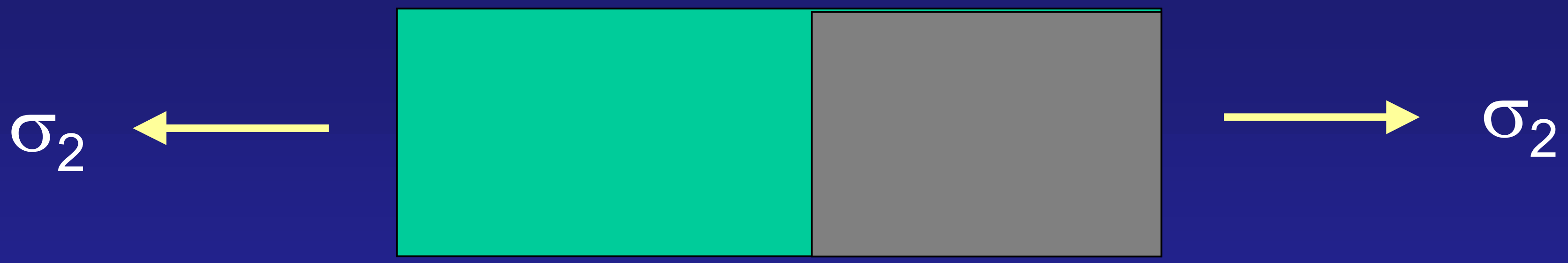
Unidirectional ply - transverse tensile modulus

For the transverse stiffness, a load is applied at right angles to the fibres.

The model is very much simplified, and the fibres are lumped together:



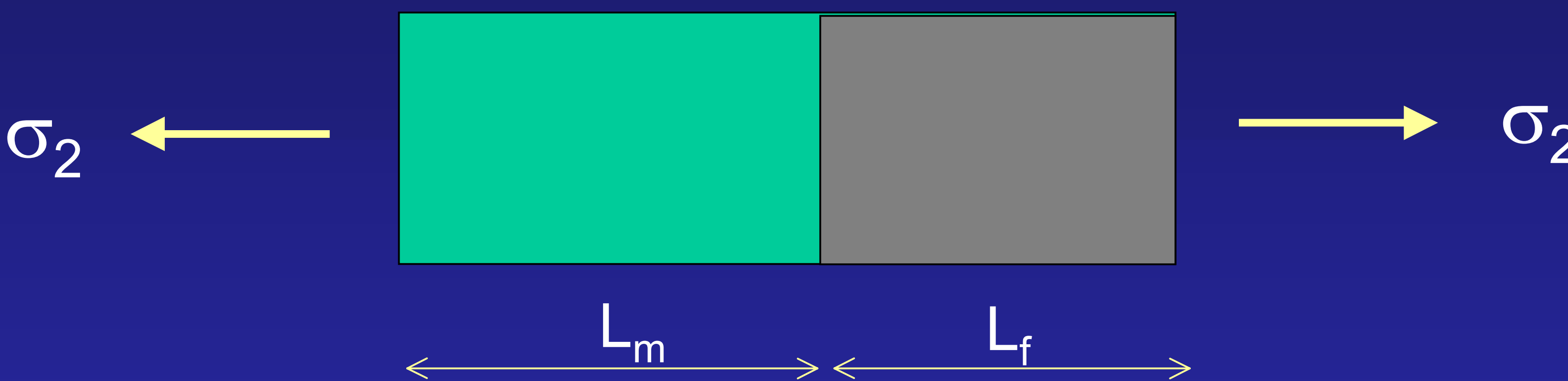
Unidirectional ply - transverse tensile modulus



It is assumed that the stress is the same in each component ($\sigma_2 = \sigma_f = \sigma_m$).

Poisson contraction effects are ignored.

Unidirectional ply - transverse tensile modulus

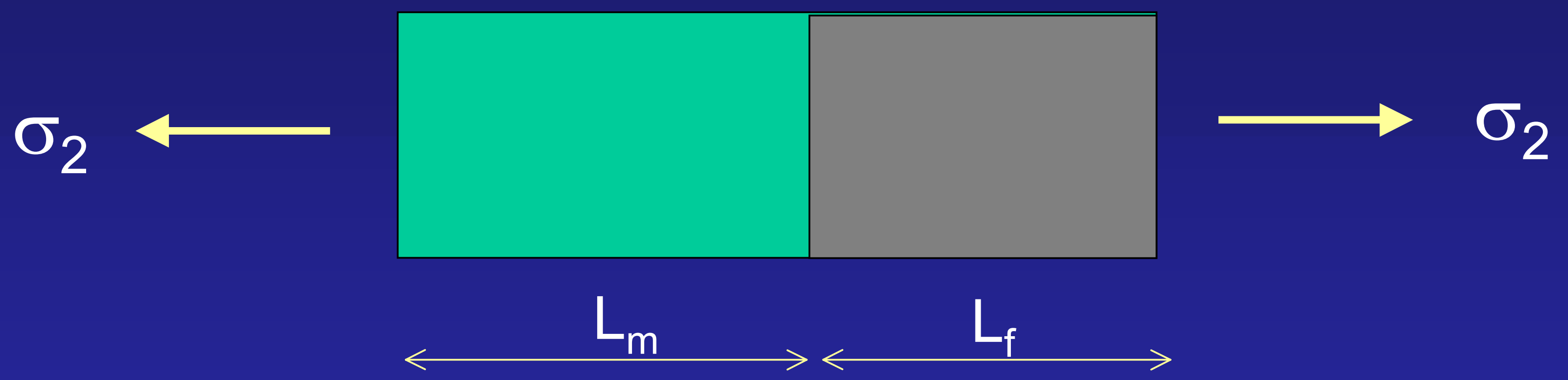


The total extension is $\delta_2 = \delta_f + \delta_m$, so the strain is given by:

$$\varepsilon_2 L_2 = \varepsilon_f L_f + \varepsilon_m L_m$$

so that $\varepsilon_2 = \varepsilon_f\,(L_f / L_2) + \varepsilon_m\,(L_m / L_2)$

Unidirectional ply - transverse tensile modulus

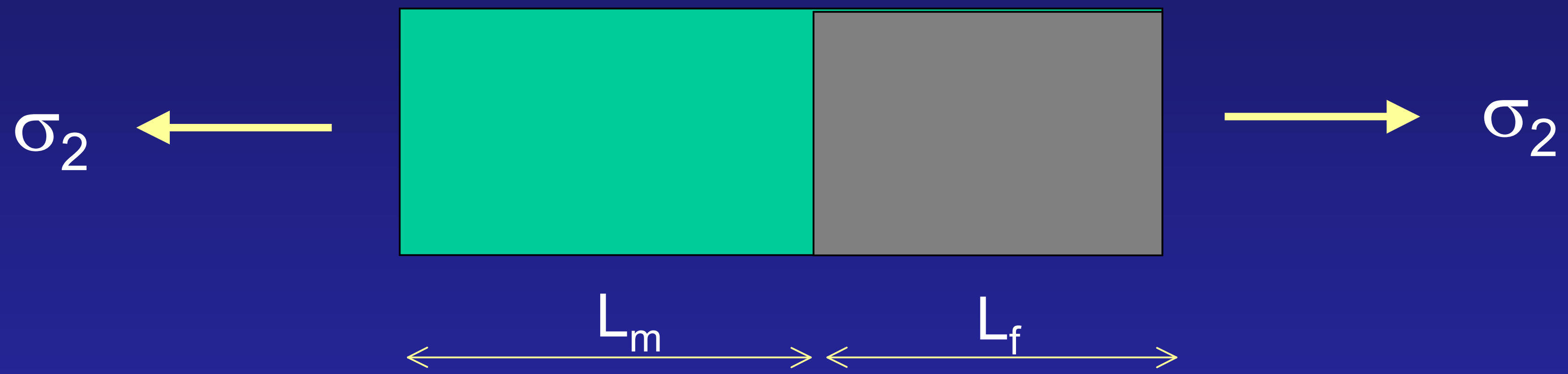


But $L_f / L_2 = V_f$ and $L_m / L_2 = V_m = 1\text{-}V_f$

So $\varepsilon_2 = \varepsilon_f V_f + \varepsilon_m (1\text{-}V_f)$

and $\sigma_2 / E_2 = \sigma_f V_f / E_f + \sigma_m (1\text{-}V_f) / E_m$

Unidirectional ply - transverse tensile modulus

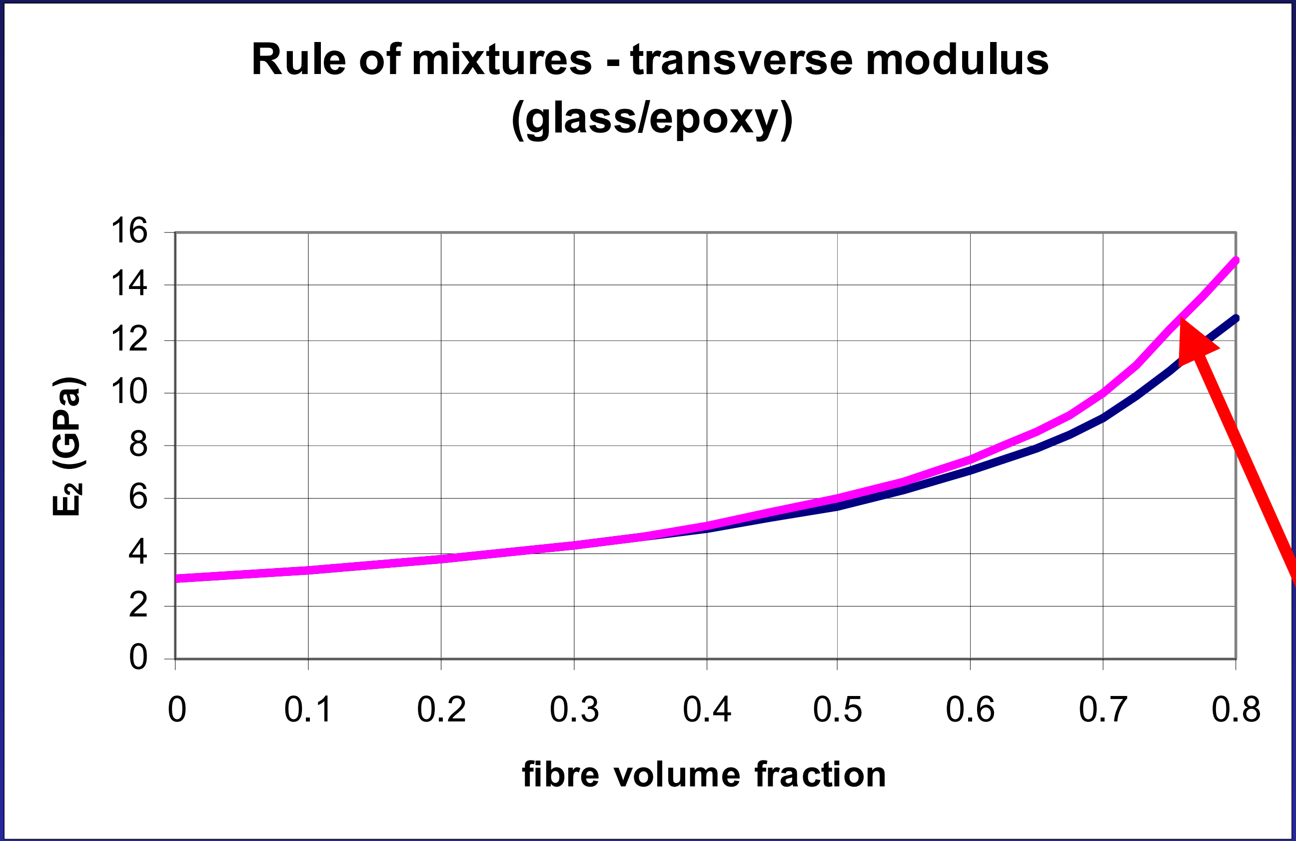


But $\sigma_2 = \sigma_f = \sigma_m$, so that:

$$\frac{1}{E_2} = \frac{V_f}{E_f} + \frac{(1 - V_f)}{E_m}$$

or

$$E_2 = \frac{E_f E_m}{E_m V_f + E_f (1 - V_f)}$$

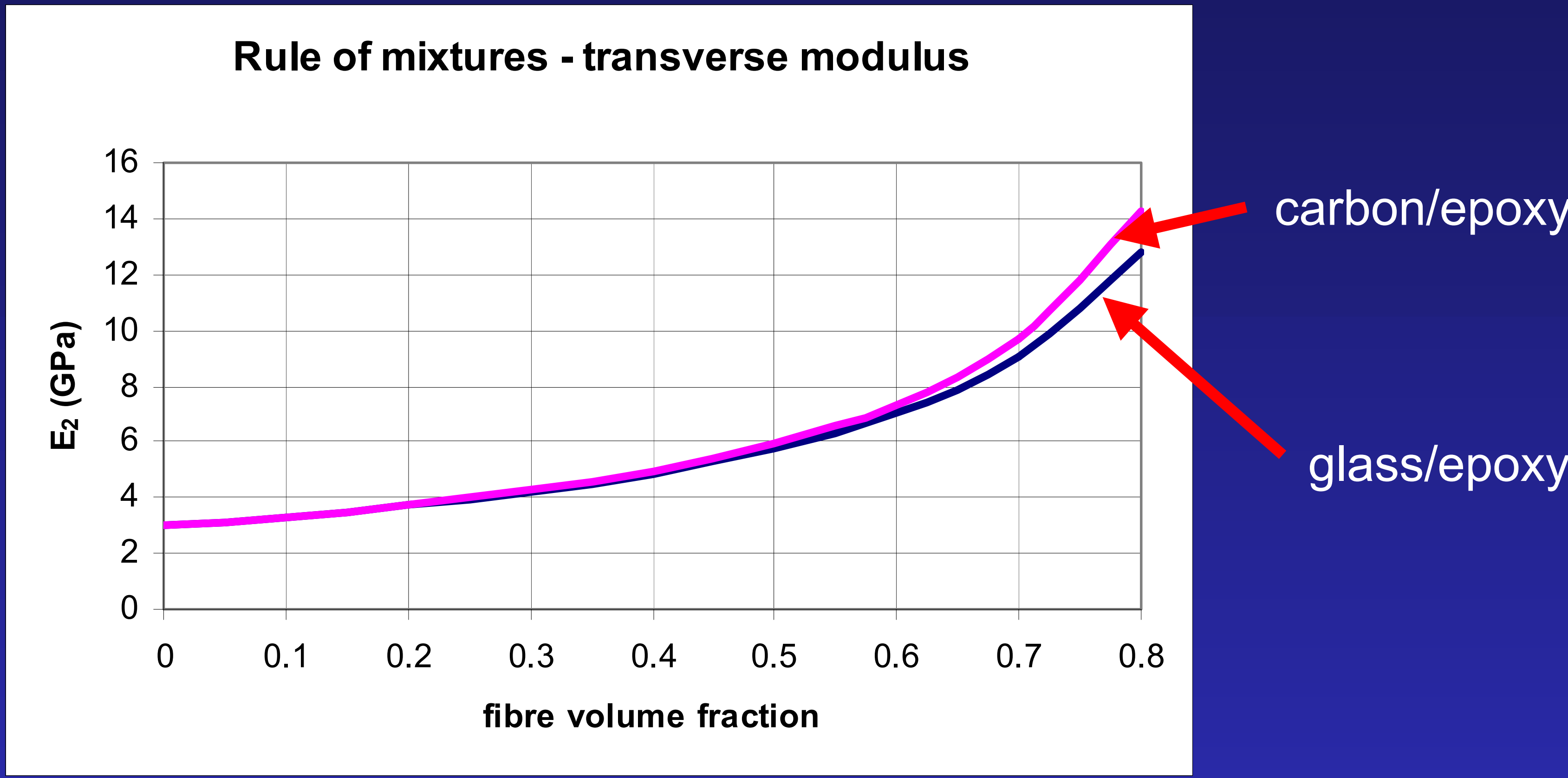


If $E_f >> E_m$,

$E_2 \approx E_m / (1-V_f)$

Note that E_2 is not particularly sensitive to V_f.

If $E_f >> E_m$, E_2 is almost independent of fibre property:



The transverse modulus is dominated by the matrix, and is virtually independent of the reinforcement.

The transverse rule of mixtures is not particularly accurate, due to the simplifications made - Poisson effects are not negligible, and the strain distribution is not uniform:

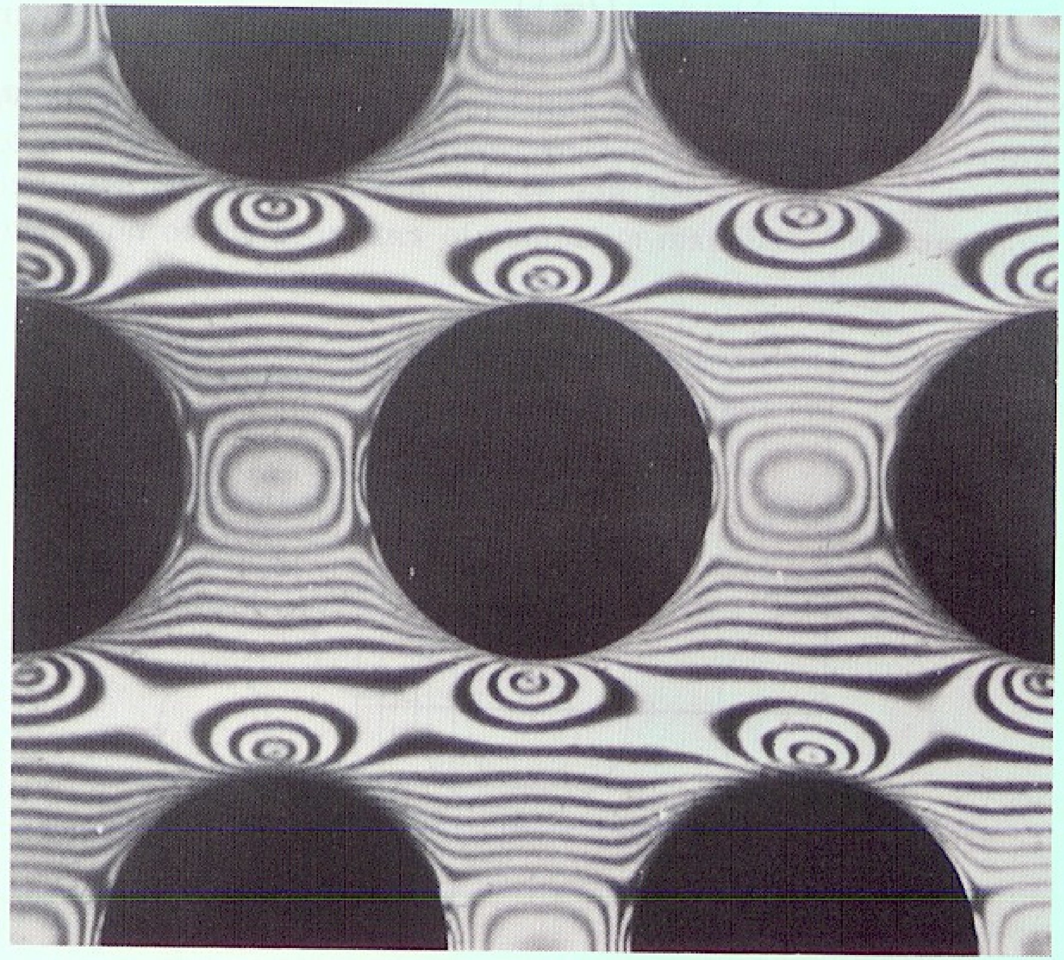


Fig. 5.8. Isochromatic fringes in a macromodel composite material loaded in transverse tension. From Puck (1967).

(source: Hull, Introduction to Composite Materials, CUP)

Unidirectional ply - transverse tensile modulus

Many theoretical studies have been undertaken to develop better micromechanical models (eg the semi-empirical Halpin-Tsai equations).

A simple improvement for transverse modulus is

$$E_2 = \frac{E_f E'_m}{E'_m V_f + E_f(1 - V_f)}$$

where

$$E'_m = \frac{E_m}{1 - \nu_m^2}$$

Generalised rule of mixtures for tensile modulus

$$E = \eta_L \eta_o E_f V_f + E_m (1-V_f)$$

η_L is a length correction factor. Typically, $\eta_L \approx 1$ for fibres longer than about 10 mm.

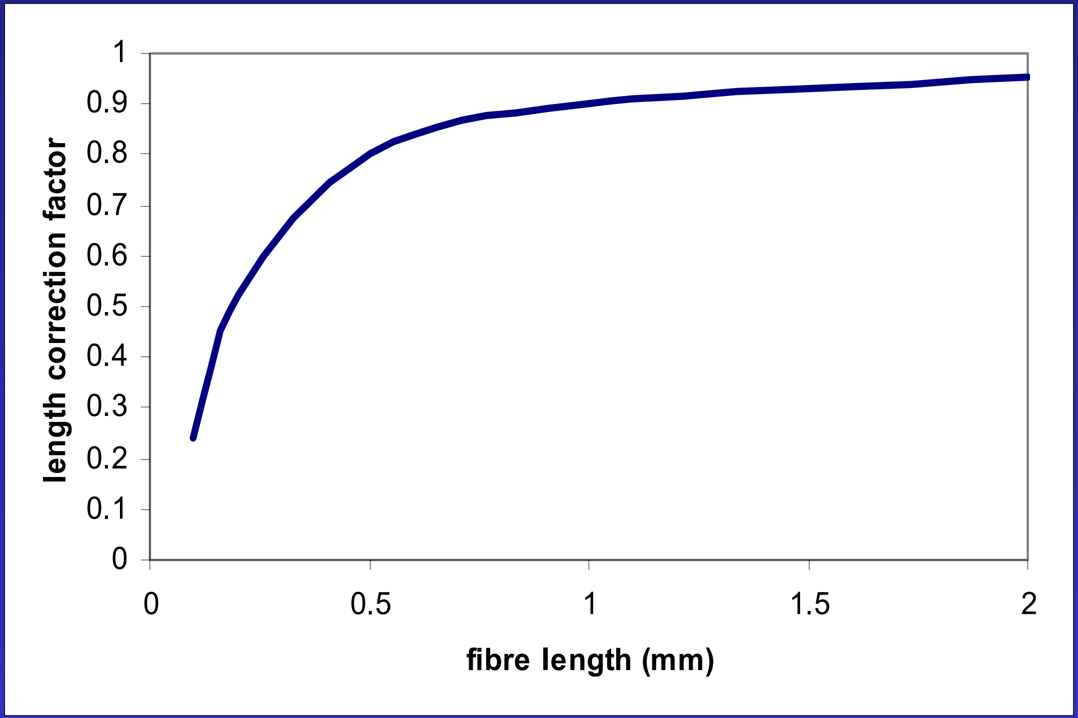
η_o corrects for non-unidirectional reinforcement:

	η_o
unidirectional	1.0
biaxial	0.5
biaxial at $\pm 45^o$	0.25
random (in-plane)	0.375
random (3D)	0.2

Theoretical length correction factor

$$\eta_L = 1 - \frac{\tanh(\beta L/2)}{(\beta L/2)}$$

$$\beta = \sqrt{\frac{8G_m}{E_f D^2 \ln(2R/D)}}$$



Theoretical length correction factor for glass fibre/epoxy, assuming inter-fibre separation of 20 D.

Stiffness of short fibre composites

For aligned short fibre composites (difficult to achieve in polymers!), the rule of mixtures for modulus in the fibre direction is:

$$E = \eta_L E_f V_f + E_m(1 - V_f)$$

The length correction factor (η_L) can be derived theoretically. Provided L > 1 mm, η_L > 0.9

For composites in which fibres are not perfectly aligned the full rule of mixtures expression is used, incorporating both η_L *and* η_o.

In short fibre-reinforced thermosetting polymer composites, it is reasonable to assume that the fibres are always well above their critical length, and that the elastic properties are determined primarily by orientation effects.

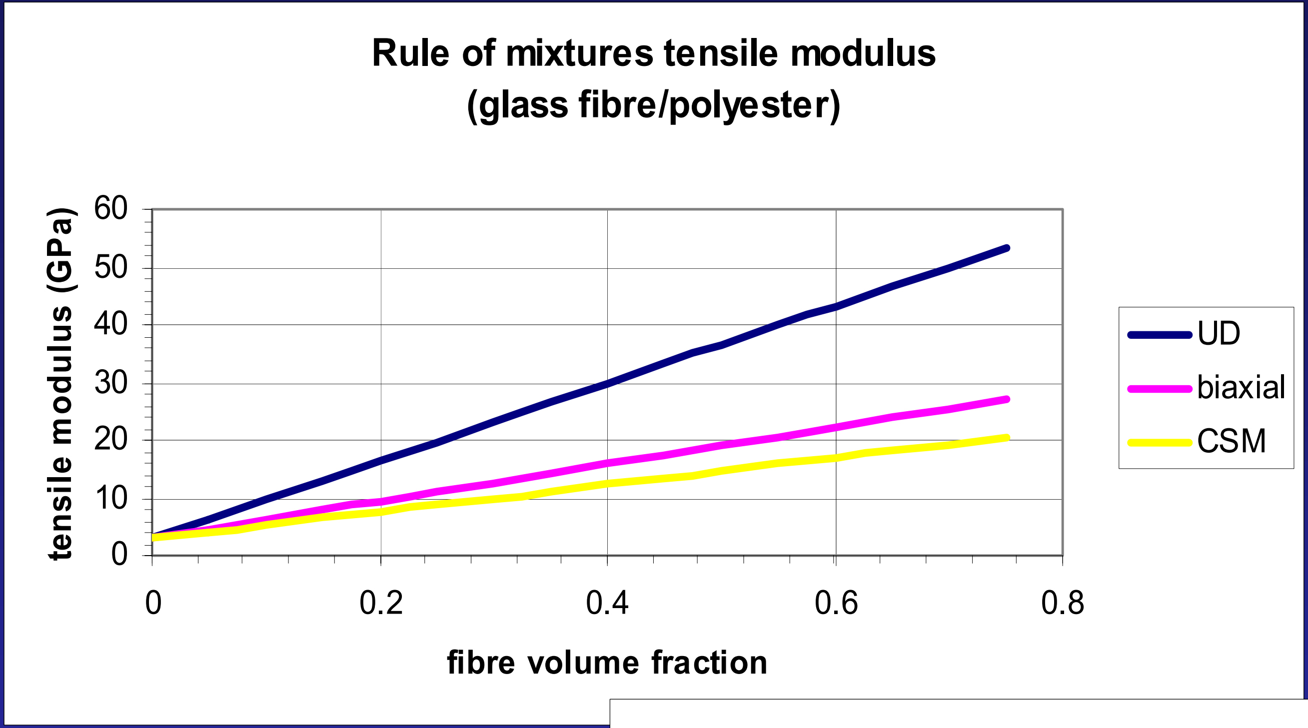
The following equations give reasonably accurate estimates for the isotropic in-plane elastic constants:

$$E = \frac{3}{8}E_1 + \frac{5}{8}E_2$$

$$G = \frac{1}{8}E_1 + \frac{1}{4}E_2$$

$$\nu = \frac{E}{2G} - 1$$

where E_1 and E_2 are the ‘UD’ values calculated earlier



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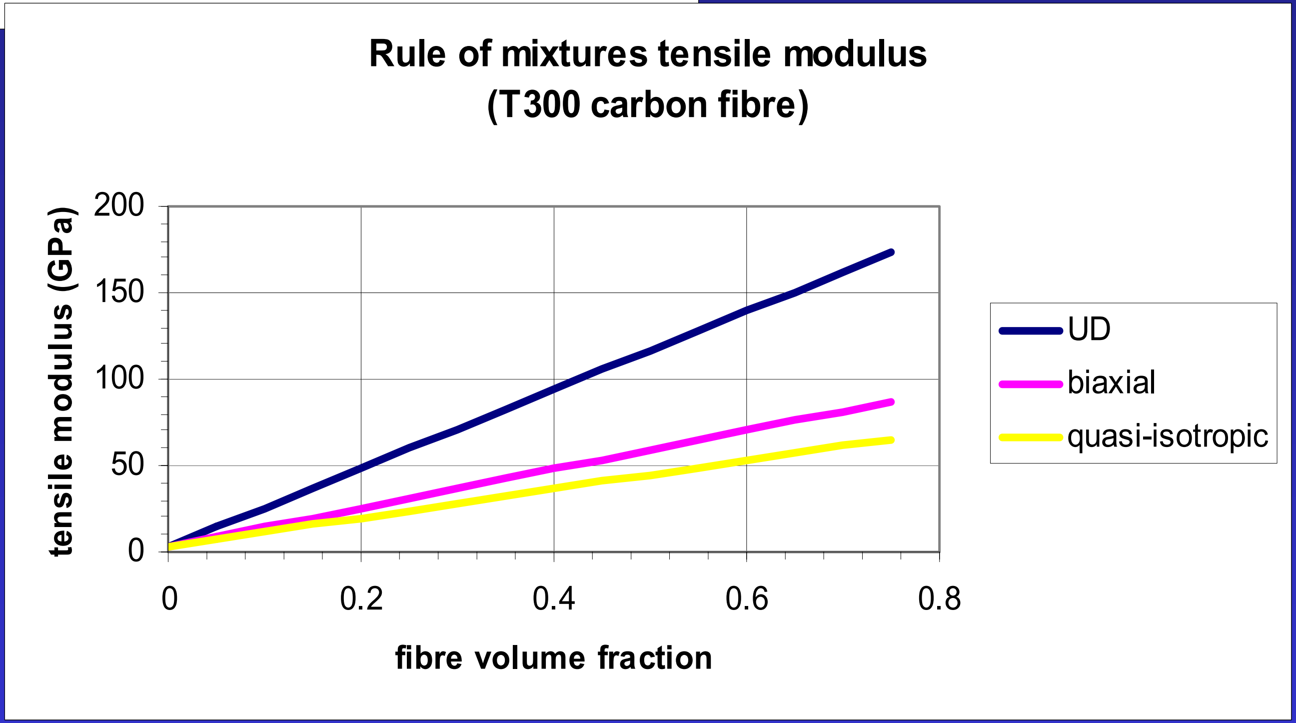

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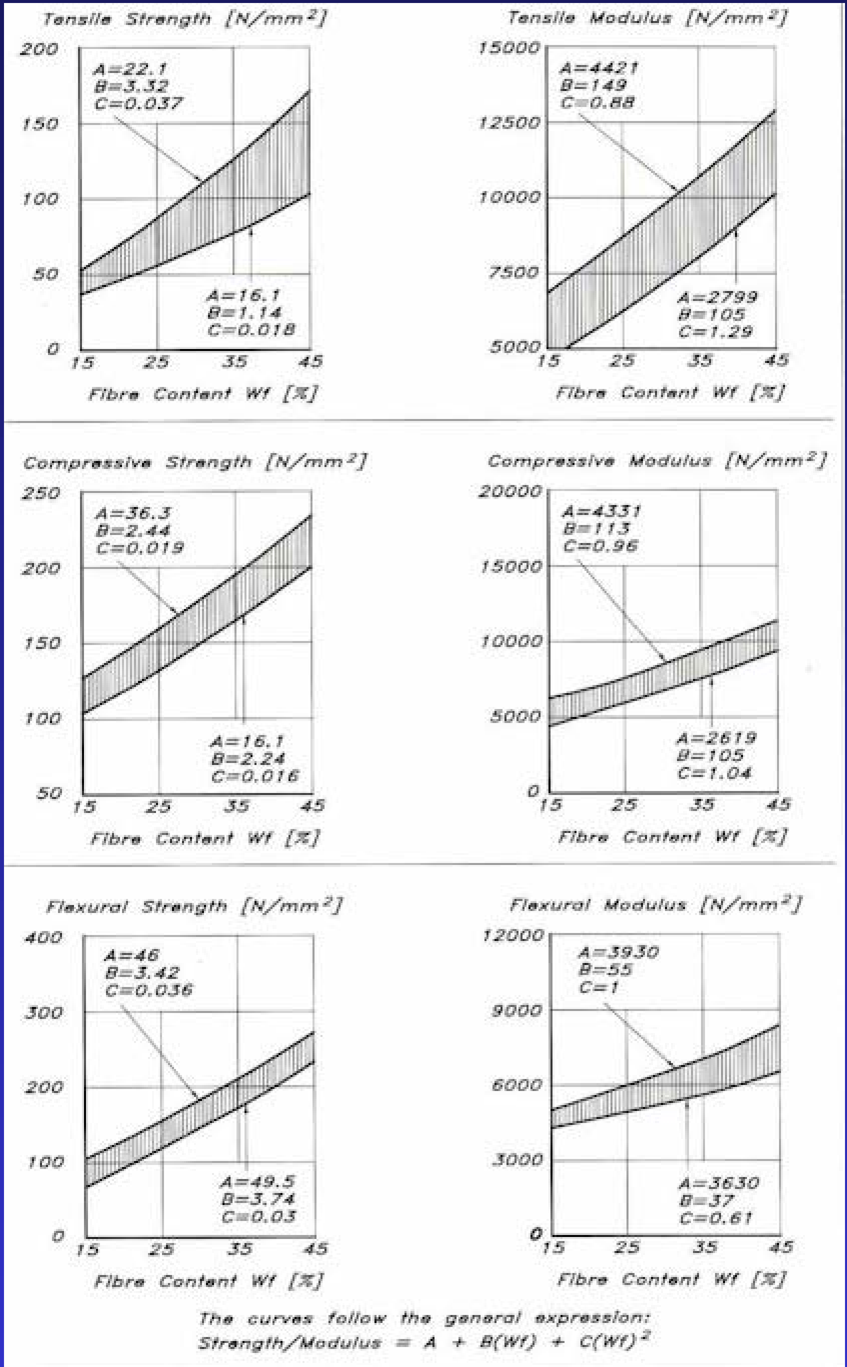
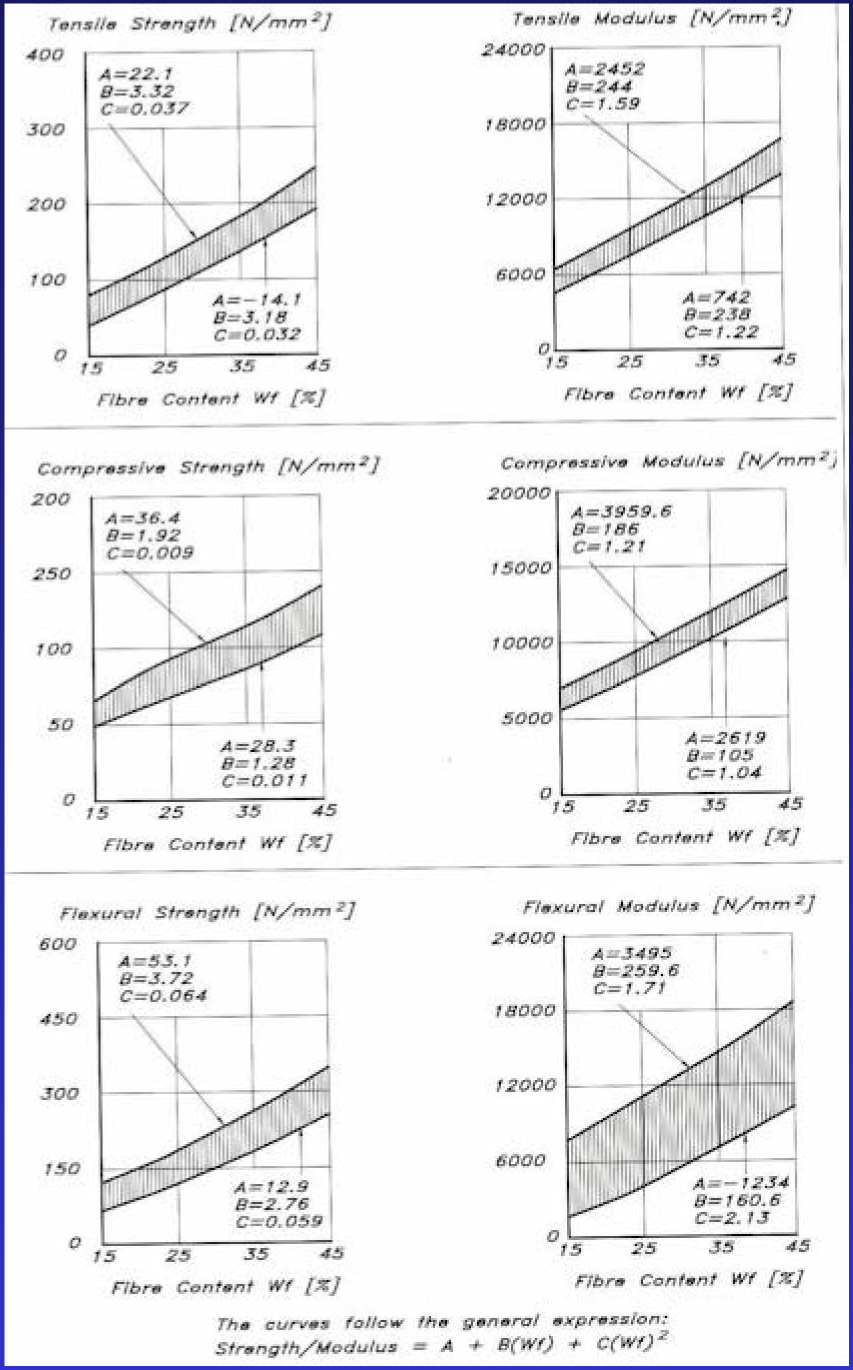


Rules of mixture properties for CSM-polyester laminates

Larsson & Eliasson, Principles of Yacht Design

Rules of mixture properties for glass woven roving-polyester laminates

Larsson & Eliasson, Principles of Yacht Design



Other rules of mixtures

- Shear modulus:

$$\frac{1}{G_{12}} = \frac{V_f}{G_f} + \frac{(1-V_f)}{G_m}$$

- Poisson's ratio:

$$\nu_{12} = \nu_f V_f + \nu_m (1-V_f)$$

- Thermal expansion:

$$\alpha_1 = \frac{1}{E_1}(\alpha_f E_f V_f + \alpha_m E_m [1-V_f])$$

$$\alpha_2 = \alpha_f V_f (1+\nu_f) + \alpha_m (1-V_f)(1+\nu_m) - \alpha_1 \nu_{12}$$