3.6. Bernoulli's equation

Bernoulli's equation is a special case of the steady flow energy equation:

$$q + w_s = \frac{1}{2}(c_2^2 - c_1^2) + g(z_2 - z_1) + (h_2 - h_1)$$

Remember the key assumptions: steady flow

properties constant with time.

Consider the special case of an **incompressible** fluid (constant density), and **isothermal** (no change in temperature), **frictionless** and **adiabatic** flow. This might sound like a lot of restrictions, but in practice many instances of pipe flow approximate very closely to these conditions.

So, under these conditions, $q = 0$, and $w_s = 0$, and Δu (the change in internal energy due to changes in temperature) = 0.

The equation becomes:

$$0 = \frac{1}{2}(c_2^2 - c_1^2) + g(z_2 - z_1) + (h_2 - h_1)$$

But $h_2 - h_1 = u_2 + p_2 v_2 - u_1 - p_1 v_1$

so that, if $u_2 = u_1$, $h_2 - h_1 = p_2 v_2 - p_1 v_1$

This can also be written as $\frac{p_2}{\rho_2} - \frac{p_1}{\rho_1}$ (because $v = \frac{1}{\rho}$)

But for an incompressible fluid, $\rho_2 = \rho_1 = \rho$

Thus, the SFEE becomes, under these conditions,

$$0 = \frac{1}{2}(c_2^2 - c_1^2) + g(z_2 - z_1) + \frac{p_2}{\rho} - \frac{p_1}{\rho}$$

It is normally written, dividing all through by g, and collecting terms at each flow station together:

$$\boxed{\frac{p_1}{\rho g} + \frac{c_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{c_2^2}{2g} + z_2}$$

This is **Bernoulli's equation**.

What are the units of each term in the equation above?

If the equation is multiplied all through by ρg, it becomes:

$$p_1 + \frac{1}{2}\rho c_1^2 + \rho g z_1 = p_2 + \frac{1}{2}\rho c_2^2 + \rho g z_2$$

which is the same equation, **Bernoulli's equation**, expressed in terms of **pressure**. You may remember we talked of "pressure energy". We started with the steady flow energy equation, and we have ended up with an equation in pressure. The pressures can be thought of as an "energy per unit volume".

Check that energy/volume (Jm^{-3}) has the same units as pressure.

Pressure terms

The pressure terms in Bernoulli's equation have special names:

p is called the **static pressure**, and is the pressure that would be measured by a pressure gauge travelling along with the fluid, or the pressure in a bubble moving with the flow;

$\frac{1}{2}\rho c^2$ is called the **dynamic pressure**, and is the pressure due the fluid velocity, the kinetic energy of the fluid per unit volume

ρgz is called the **potential pressure**, and is the pressure the fluid possesses due to its height above a datum, its potential energy per unit volume.

The static and dynamic pressures taken together are called **the stagnation pressure**, p_{st}, which is the pressure realised when a flowing fluid is brought to rest.

$$p_{st} = p + \frac{1}{2}\rho c^2$$

Friction losses

To account for the fact that there is normally friction in pipes, the equation may be modified. The modified Bernoulli equation is:

In terms of "**head**", measured in m:

$$\frac{p_1}{\rho g} + \frac{c_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{c_2^2}{2g} + z_2 + h_{losses}$$

or in terms of **pressure**:

$$p_1 + \frac{1}{2}\rho c_1^2 + \rho g z_1 = p_2 + \frac{1}{2}\rho c_2^2 + \rho g z_2 + \Delta p_{dat}$$

Δp_{dat} represents the pressure drop due to frictional losses, either internal due to the viscosity of the fluid, or external such as due to surface roughness of the pipe. Similarly, h_{losses} represents the loss of head due to the same factors.

Pumps and turbines

Bernoulli's equation is applied between any two flow stations within the flow. Between the two selected flow stations there may be either a pump or a turbine installed. These can be accounted for in Bernoulli's equation. A pump will increase the energy of the fluid, and therefore, the pressure drop across the pump will appear on the left hand side of Bernoulli's equation:

$$p_1 + \frac{1}{2}\rho c_1^2 + \rho g z_1 + \Delta p_{pump} = p_2 + \frac{1}{2}\rho c_2^2 + \rho g z_2$$

A turbine however, extracts energy from the fluid, and therefore, the pressure drop across a turbine appears on the right hand side of the equation.

$$p_1 + \frac{1}{2}\rho c_1^2 + \rho g z_1 = p_2 + \frac{1}{2}\rho c_2^2 + \rho g z_2 + \Delta p_{turb}$$

OR

Another way to look at this is to consider the **rate of working** either to the pump or from the turbine.

Power – rate of energy transfer, or rate of doing work is $\dot{\mathbf{W}}$, measured in watts.
If the mass flow rate is $\dot{\mathbf{m}}$ then the energy transfer per unit mass is

$$w = \frac{\dot{W}}{\dot{m}}$$

So the steady flow **energy** equation becomes:

$$\frac{p_1}{\rho} + \frac{c_1^2}{2} + g z_1 + \frac{\dot{W}_{pump}}{\dot{m}} = \frac{p_2}{\rho} + \frac{c_2^2}{2} + g z_2 + \frac{\dot{W}_{turbine}}{\dot{m}} + g h_{losses}$$

or, in terms of **head**:

$$\frac{p_1}{\rho g} + \frac{c_1^2}{2g} + z_1 + h_{pump} = \frac{p_2}{\rho g} + \frac{c_2^2}{2g} + z_2 + h_{turbine} + h_{losses}$$

where $h_{pump} = \frac{\dot{W}_{pump}}{\dot{m}g}$ and $h_{turbine} = \frac{\dot{W}_{turbine}}{\dot{m}g}$

NOTE that $\dot{W}_{pump} = \dot{V}(\Delta p)_{pump}$ where $\dot{V}$ is the volume flow rate.

Further reading:

Massey, Mechanics of fluids 3.5-3.6

The Open University, T236 Introduction to thermofluid mechanics Block 2