

Engineering Science 3 - MECH 226

(c) Damped forced vibration – applied force

Let us see what happens if we now introduce viscous damping into the system. As in the case of free vibration, a force proportional to the velocity, which opposes the motion is introduced as shown in Figure 2.7. This shows a damped single degree-of-freedom system for the case when the excitation is by a sinusoidal force applied to the mass.

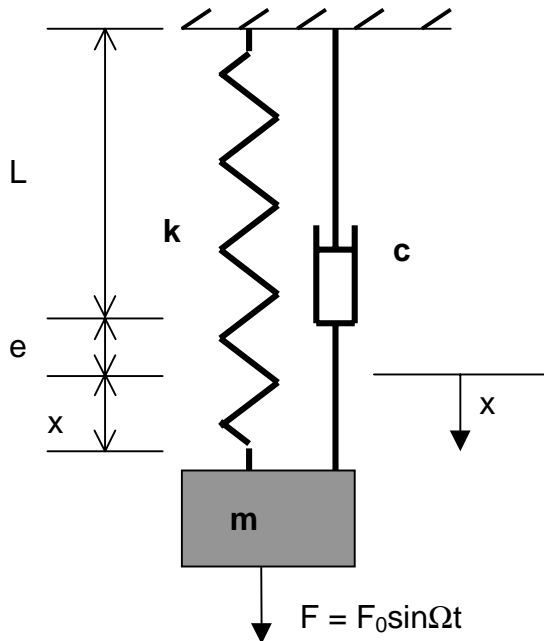


Figure 2.7. Damped single DOF system undergoing forced vibration by an applied force.

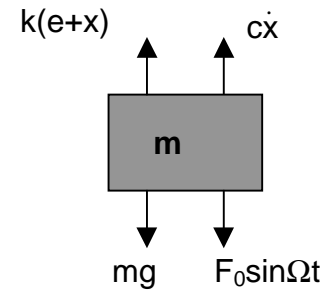


Figure 2.8. FBD of weight suspended by a spring, with damping

Applying Newton's second law in the usual way, using the free body diagram in Figure 2.8, we find that the equation of motion is like that for the undamped case, but with the viscous damping force included:

$$F_0 \sin \Omega t = m\ddot{x} + c\dot{x} + kx$$

which yields
$$\ddot{x} + 2\zeta\omega\dot{x} + \omega^2 x = \frac{F_0}{m} \sin \Omega t \quad (2.5)$$

where, as before, ω is the natural frequency, $\sqrt{\frac{k}{m}}$ and $\zeta = \frac{c}{2\sqrt{km}}$.

Adopting the same reasoning as before, we look for a particular solution (to represent the steady state vibration of the mass) of the form $x = X_0 \sin(\Omega t - \phi)$.

It can be shown (see Appendix 1) that this is indeed a solution if:

$$X_0 = \frac{\omega^2 F_0 / k}{\sqrt{(\omega^2 - \Omega^2)^2 + (2\zeta\omega\Omega)^2}} = \frac{F_0 / k}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} \quad (2.6)$$

and
$$\tan \phi = \frac{2\zeta\omega\Omega}{\omega^2 - \Omega^2} = \frac{2\zeta r}{1 - r^2} \quad (2.7)$$

The way that the response amplitude X_0 varies with the driving frequency is shown in Figure 2.9 where X_0 is plotted against the non-dimensional frequency r , which is directly proportional to the driving frequency. In Figure 2.10 the phase angle ϕ is plotted against r , where the phase angle has been calculated in degrees. It can also be measured in radians (π radians = 180°). Curves are shown for different values of ζ to illustrate the effect on the response of the amount of damping in the system.

These results show up some interesting trends, which are borne out in practice. Firstly, we can see that the maximum response amplitude depends on how much damping there is in the system. The less damping there is (i.e. the smaller is ζ) the larger it gets, and if there is no damping at all, then the maximum amplitude possible is infinite, as we saw earlier.

Figure 2.9. The response amplitude of a damped system with an applied force as a function of the frequency ratio

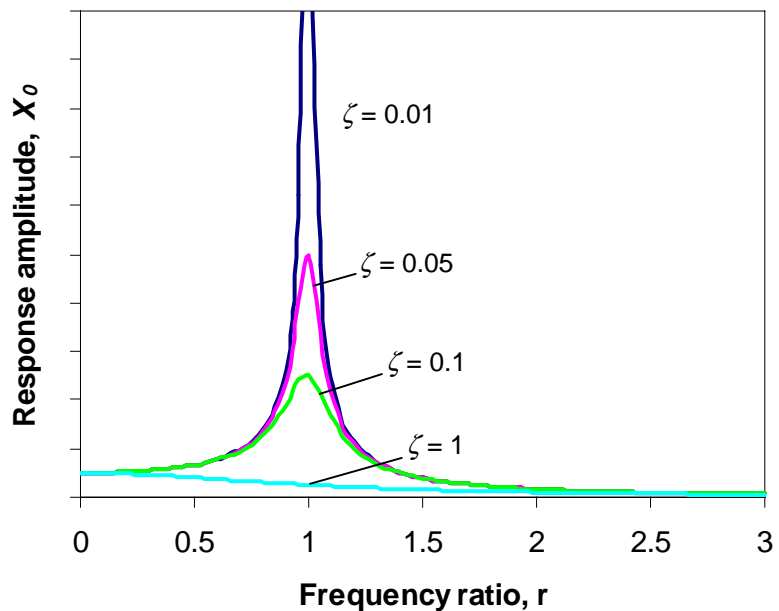
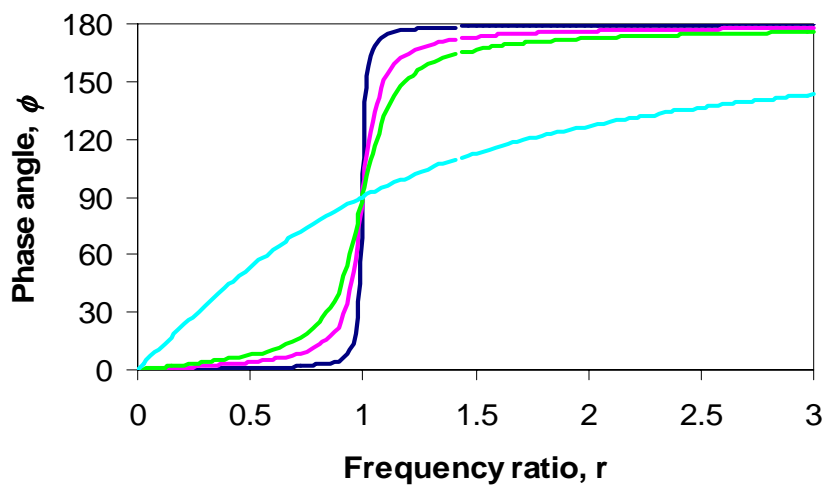


Figure 2.10. The phase angle of a damped system with an applied force as a function of the frequency ratio



It is clear, too, that the more damping there is, the more slowly does the amplitude build up to its maximum as the driving frequency is increased. At low damping, there is a sudden increase in amplitude as the driving frequency approaches the natural frequency, and a sudden drop in amplitude as r becomes bigger than 1. While, as the damping is increased, the build up to the maximum amplitude is much more gradual, and “resonance” becomes less sharp.

We can see that at very low driving frequencies (small r), the amplitude of the response is almost the same as the static displacement, F_0/k , but at very high driving frequencies (large r), the amplitude tends to become zero, no matter how little damping there is. The system simply doesn't have time to respond.

You may not be able to see very clearly from Figure 2.9 that the maximum amplitude of the response does not occur when $r = 1$, as it did for the undamped system. From equation (2.6) it should be clear that X_0 will be a maximum when the denominator $[(1-r^2)^2 + (2\zeta r)^2]$ is a minimum. To find the value of r at which this occurs, we need to differentiate with respect to r (see Appendix 1). We find that the amplitude becomes a maximum when $r = \sqrt{1-2\zeta^2}$ or $\Omega = \omega\sqrt{1-2\zeta^2}$. For small ζ , the maximum occurs when $r \approx 1$, but if the system has high damping, then the frequency at which the response becomes a maximum is lower than the natural frequency.

Once again, we see that a driving frequency near to the natural frequency produces rather critical conditions, and if excessive amplitudes are to be avoided, then driving the system close to its resonant frequency is to be avoided. This means that either the system should be driven at a different frequency or the stiffness, k , or mass, m , should be altered to change the natural frequency, ω .

We can observe from Figure 2.10 that when the driving frequency is low, the response is very nearly in phase with the excitation, shown by a small value of ϕ , but when the frequency is high, the response is 180° out of phase (exactly out of phase), while at $r = 1$, when the driving frequency is exactly the same as the natural frequency, the response is 90° out of phase. So, as the frequency is gradually increased, the response will change from being in phase at frequencies below the resonant frequency to being out of phase at frequencies above it. If the damping in the system is low, this change is very sudden, and can readily be observed experimentally. As the damping is increased, the change becomes more gradual. With critical damping ($\zeta = 1$), the change is very gradual, and the response becomes exactly out of phase only at very high driving frequencies.

Note that the term “resonant frequency” is used sometimes to denote the frequency at which the response amplitude becomes a maximum, and sometimes the frequency at which $r = 1$. For low damping, the two are very nearly the same, but for highly damped systems it should be made clear which definition is used. In this course, I will stick to the definition of resonant frequency as that for which $r = 1$, so that the resonant frequency and the undamped natural frequency are the same, while the frequency at which the amplitude becomes a maximum is $\omega\sqrt{1-2\zeta^2}$.

The magnification, M , as in the previous section, is:

$$M = \frac{X_0}{F_0/k} = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

So a plot of M against r will give the same result as shown in Figure 2.9 if $F_0/k = 1$.

The transmissibility, defined above, $T = P_0/F_0$, where P_0 is the amplitude of the force transmitted to the support. With viscous damping, the force transmitted is

$$P = c\dot{x} + kx = c\Omega X_0 \cos(\Omega t - \phi) + kX_0 \sin(\Omega t - \phi)$$

This can be shown (see Appendix 1) to be of the form: $P = P_0 \sin(\Omega t - \phi + \delta)$

where $P_0 = F_0 \left[\frac{1 + (2\zeta r)^2}{(1-r^2)^2 + (2\zeta r)^2} \right]^{1/2}$ and $\tan \delta = 2\zeta r$

The transmissibility T is, therefore $T = \left[\frac{1 + (2\zeta r)^2}{(1-r^2)^2 + (2\zeta r)^2} \right]^{1/2}$

This looks rather a complicated expression, but once you know ζ and r , it just needs a bit of care putting the numbers into your calculator to come up with the right answer. The way the transmissibility varies with r is shown in Figure 2.11.

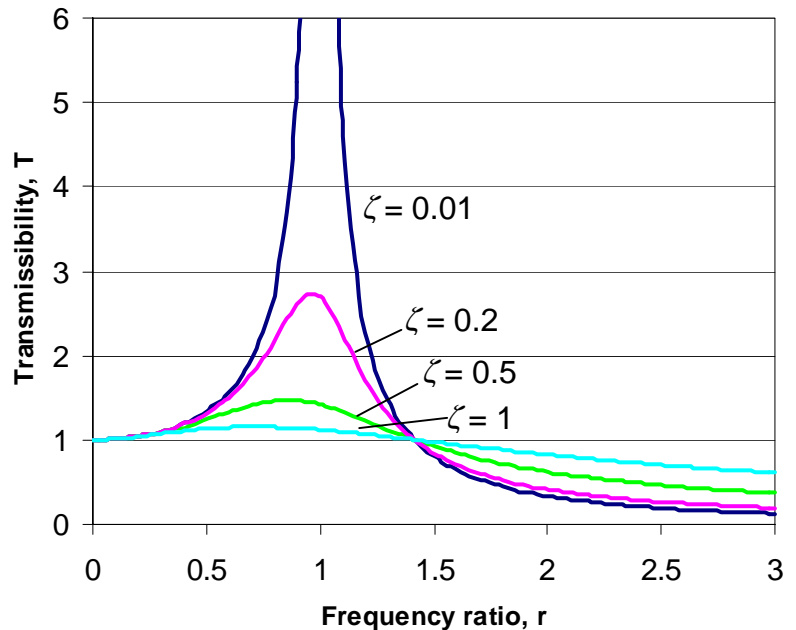


Figure 2.11. The transmissibility of a damped single DOF system as a function of the frequency ratio.

As with the maximum response amplitude, the force transmitted to the support becomes a maximum when $\Omega = \omega\sqrt{1-2\zeta^2}$. The force transmitted is equal to the driving force at $\Omega = \sqrt{2}\omega$. As the damping in the system is increased, the magnitude of the force transmitted gets smaller as long as $\Omega < \sqrt{2}\omega$, but when

the drive frequency is higher than this, the effect of increasing the damping is reversed.

(d) Damped forced vibration – ground excitation

You should be familiar now with the way to set about looking at these problems, so Figure 2.12 which shows a damped single degree-of-freedom system undergoing forced vibration by means of ground excitation should come as no surprise to you. Figure 2.13 shows the free body diagram of the mass, and the equation of motion is derived exactly in the way we have used before, by applying Newton’s second law. The results are given below.

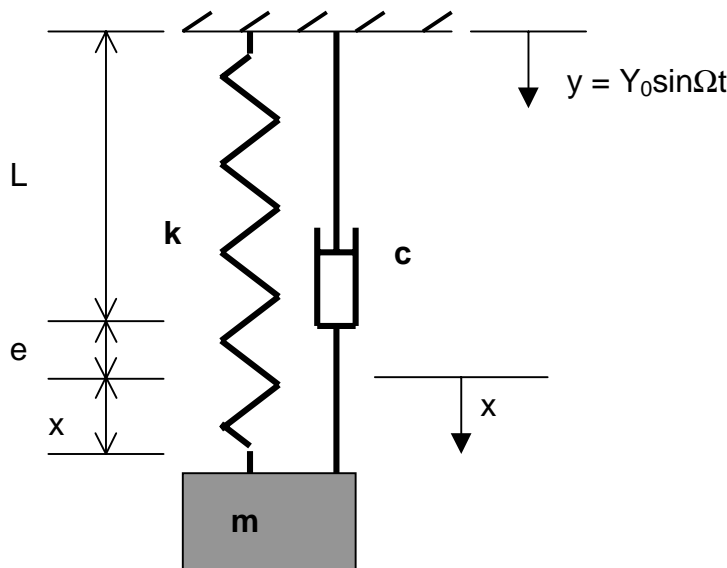


Figure 2.12. Forced vibration of a damped single DOF system by ground excitation

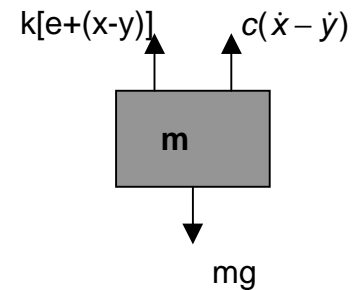


Figure 2.13. FBD of weight suspended by a spring, with damping

The equation of motion: $\Sigma F_x = mg - k[e + (x - y)] - c(\dot{x} - \dot{y}) = m\ddot{x}$

$$\therefore ky + c\dot{y} = m\ddot{x} + c\dot{x} + kx$$

Dividing all through by m , and rewriting:

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = \frac{k}{m}Y_0 \sin \Omega t + \frac{c\Omega}{m}Y_0 \cos \Omega t$$

$$\therefore \ddot{x} + 2\zeta\omega\dot{x} + \omega^2x = \omega^2Y_0 \sin \Omega t + 2\zeta\omega\Omega Y_0 \cos \Omega t \tag{2.8}$$

where, as before, ω is the natural frequency, $\sqrt{\frac{k}{m}}$ and $\zeta = \frac{c}{2\sqrt{km}}$.

It can be shown (see Appendix 1) that this has the particular solution:

$$x = X_0 \sin(\Omega t - \psi)$$

$$\text{where } X_0 = Y_0 \left[\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2} \right]^{1/2} \quad \text{and } \tan \psi = \frac{2\zeta r^3}{1 - r^2 + (2\zeta r)^2} \quad (2.9)$$

So, the transmissibility, $T = \frac{X_0}{Y_0} = \left[\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2} \right]^{1/2}$ is exactly the same for both types of forced vibration.

Damped forced vibration produces a similar response, therefore, whether the excitation is by a sinusoidal force applied to the mass or by sinusoidal motion of the ground. Remember that initial free vibration dies out. This is known as transient vibration. It is the steady state response that is interesting.

Summary: Forced vibration with viscous damping

For a simple damped spring-mass system, undergoing forced vibration by means of a sinusoidal force $F_0 \sin \Omega t$, the motion of the mass is sinusoidal and of the same frequency as the excitation.

The steady state response is $x = X_0 \sin(\Omega t - \phi)$

$$\text{where } X_0 = \frac{\omega^2 F_0 / k}{\sqrt{(\omega^2 - \Omega^2)^2 + (2\zeta \omega \Omega)^2}} = \frac{F_0 / k}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}$$

$$\text{and } \tan \phi = \frac{2\zeta \omega \Omega}{\omega^2 - \Omega^2} = \frac{2\zeta r}{1 - r^2}$$

The maximum response amplitude occurs when $\Omega = \omega \sqrt{1 - 2\zeta^2}$ and is

$$\text{equal to } X_{0\max} = \frac{F_0 / k}{2\zeta \sqrt{1 - \zeta^2}}$$

$$\text{The magnification, } M = \frac{X_0}{F_0 / k} = \frac{1}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}$$

The force transmitted to the support is $P = P_0 \sin(\Omega t - \phi + \delta)$

where $P_0 = T F_0$, T is the transmissibility and $\tan \delta = 2\zeta r$

$$T = \left[\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2} \right]^{1/2}$$

If the forced vibration is due to ground excitation $y = Y_0 \sin \Omega t$, the steady state response is $x = X_0 \sin(\Omega t - \psi)$

$$\text{where } X_0 = T Y_0 \quad \text{and } \tan \psi = \frac{2\zeta r^3}{1 - r^2 + (2\zeta r)^2}$$

The maximum response amplitude occurs when $\Omega = \omega \sqrt{1 - 2\zeta^2}$

The effect of damping is to limit the maximum response amplitude and to reduce the sharpness of resonance, which can be defined as occurring when the drive frequency Ω equals the natural frequency of the system, ω .

Example 4. Consider the machine of Example 3, where viscous damping has now been introduced such that damping coefficient, c is 5 kNm^{-1} . Calculate the response amplitude, the phase angle, and the force transmitted to the foundations when the driving frequency is (a) 20 Hz and (b) 2 Hz. The amplitude of the applied force is, as before, 2500 N.

Solution:

This is a case of damped forced vibration due to an applied force.

(a) As before: $m = 1000 \text{ kg}$ $F_0 = 2500 \text{ N}$ $c = 5 \text{ kNm}^{-1}$
 $\Omega = 20 \text{ Hz} = 2\pi * 20 = 125.7 \text{ rad s}^{-1}$ $k = 50 \text{ kNm}^{-1}$.

The natural frequency $\omega = 7.07 \text{ rad s}^{-1}$ and $r = 17.8$

(a) The damping ratio, $\zeta = \frac{c}{2\sqrt{km}} = \frac{5 * 10^3}{2\sqrt{50 * 10^3 * 1000}} = 0.354$

The response amplitude (equation (2.6)):

$$X_0 = \frac{F_0 / k}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = \frac{2500 / (50 * 10^3)}{\sqrt{(1-17.8^2)^2 + (2 * 0.354 * 17.8)^2}} = \frac{0.05}{\sqrt{(-315.84)^2 + 12.6^2}}$$

$$= \frac{0.05}{316.1} = 1.58 * 10^{-4} \text{ m}$$

So the response amplitude is still very small at less than 0.16 mm.

The phase angle (equation (2.7)):

$$\tan \phi = \frac{2\zeta r}{1-r^2} = \frac{2 * 0.354 * 17.8}{1-17.8^2} = \frac{12.6}{-315.84} = -0.04 \quad \therefore \phi = 177.7^\circ$$

The amplitude of the force transmitted to the foundations:

$$P_0 = F_0 T = F_0 \left[\frac{1 + (2\zeta r)^2}{(1-r^2)^2 + (2\zeta r)^2} \right]^{1/2} = 2500 * \left[\frac{1 + (2 * 0.354 * 17.8)^2}{(1-17.8^2)^2 + (2 * 0.354 * 17.8)^2} \right]^{1/2}$$

$$= 2500 * \left[\frac{1 + 12.6^2}{(-315.84)^2 + (12.6)^2} \right]^{1/2}$$

$$= 2500 * \sqrt{\frac{159.76}{99913.7}} = 100 \text{ N}$$

The force transmitted to the foundations is therefore *increased* by the introduction of damping.

(b) $\Omega = 2 \text{ Hz} = 2\pi * 2 = 12.57 \text{ rad s}^{-1}$ and $r = 1.78$

$$X_0 = \frac{F_0 / k}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = \frac{2500 / (50 * 10^3)}{\sqrt{(1-1.78^2)^2 + (2 * 0.354 * 1.78)^2}} = \frac{0.05}{\sqrt{(-2.17)^2 + 1.26^2}}$$

$$= \frac{0.05}{6.297} = 7.94 * 10^{-3} m$$

The phase angle (equation (2.7)):

$$\tan \phi = \frac{2\zeta r}{1-r^2} = \frac{2 * 0.354 * 1.78}{1-1.78^2} = \frac{1.26}{-2.16} = -0.58 \quad \therefore \phi = 149.7^\circ$$

The amplitude of the force transmitted to the foundations:

$$P_0 = F_0 T = F_0 \left[\frac{1 + (2\zeta r)^2}{(1-r^2)^2 + (2\zeta r)^2} \right]^{1/2} = 2500 * \left[\frac{1 + (2 * 0.354 * 1.78)^2}{(1-1.78^2)^2 + (2 * 0.354 * 1.78)^2} \right]^{1/2}$$

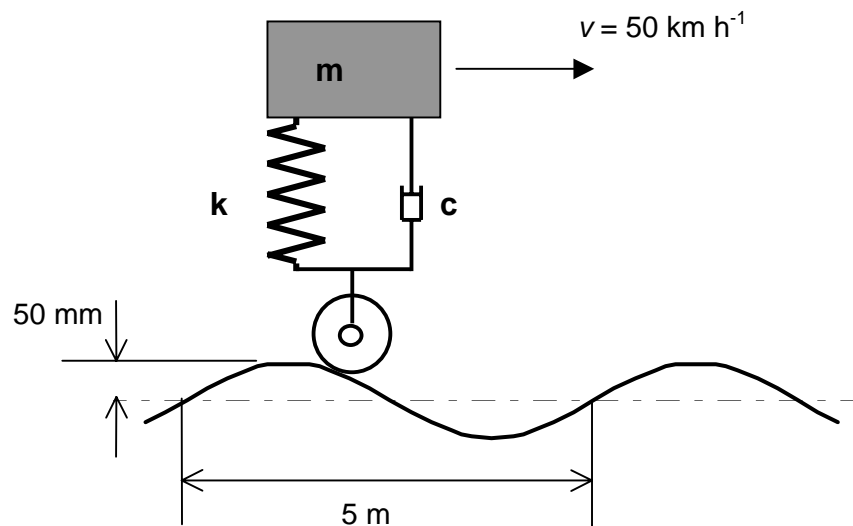
$$= 2500 * \left[\frac{1 + 1.26^2}{(-2.16)^2 + (1.26)^2} \right]^{1/2}$$

$$= 2500 * \sqrt{\frac{2.588}{6.253}} = 1608.3 N$$

The damping has reduced the response amplitude at the lower speed, but in both cases, because r is greater than $\sqrt{2}$, the force transmitted has increased with the introduction of damping.

Example 2. A trailer of mass 1000 kg is pulled with a constant speed of 50 km h^{-1} over a bumpy road which may be modelled as a sine wave of wavelength 5 m (see Figure 2.14) and amplitude 50 mm. Assume that the effective stiffness of the suspension is 350 kN m^{-1} and that the damping ratio, $\zeta = 0.5$. (a) Determine the amplitude of the motion of the trailer (b) find the speed of the trailer at which this amplitude becomes a maximum.

Figure 2.14.



Solution:

This is forced vibration due to ground excitation in effect. Imagine that the trailer is stationary and the ground underneath is being moved to the left at a speed of $50 \text{ km h}^{-1} = 50 \cdot 1000 / 3600 = 13.9 \text{ ms}^{-1}$. One cycle will have occurred when the ground has moved 5 m. This will take $5 / 13.9 \text{ s}$. So the number of cycles per second is $13.9 / 5 = 2.78$.

So the motion of the trailer over the bumpy ground is equivalent to applying a ground excitation of amplitude 50 mm and frequency 2.78 Hz. In general, the frequency will be the wavelength divided by the speed.

In this case, then, we know:

$$m = 1000 \text{ kg} \quad k = 350 \text{ kNm}^{-1} \quad \zeta = 0.5$$

$$\Omega = 2\pi f = 2\pi \cdot 2.78 = 17.47 \text{ rad s}^{-1} \quad Y_0 = 50 \text{ mm} = 0.05 \text{ m}$$

(a) First find ω and r . $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{350 \cdot 1000}{1000}} = 18.71 \text{ rad s}^{-1}$

$$\therefore r = \frac{\Omega}{\omega} = \frac{17.47}{18.71} = 0.934$$

The amplitude of the motion of the trailer (equation (2.9))

$$X_0 = Y_0 \left[\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2} \right]^{1/2} = 0.05 \left[\frac{1 + (2 \cdot 0.5 \cdot 0.934)^2}{(1 - 0.934^2)^2 + (2 \cdot 0.5 \cdot 0.934)^2} \right]^{1/2}$$

$$= 0.05 \left[\frac{1 + 0.872}{0.016 + 0.872} \right]^{1/2} = 0.05 \cdot \sqrt{2.108} = 0.073 \text{ m}$$

So the amplitude of the trailer motion is 73 mm.

(b) The maximum amplitude occurs when

$$\Omega = \omega \sqrt{1 - 2\zeta^2} = 18.71 \cdot \sqrt{1 - 2 \cdot 0.5^2} = 18.71 \cdot \sqrt{0.5} = 13.23 \text{ rad s}^{-1}$$

The driving frequency (in Hz) is the speed, v , divided by the wavelength,

$$\text{so } \Omega = \frac{2\pi \cdot v}{5} \quad \therefore v = \frac{5 \cdot \Omega}{2\pi} = \frac{5 \cdot 13.23}{2\pi} = 10.53 \text{ ms}^{-1}$$

The speed at which the amplitude of the trailer motion reaches a

$$\text{maximum is } \frac{10.53 \cdot 3600}{1000} = 38 \text{ km h}^{-1}$$

Tips on solving problems

First decide what the condition of vibration is. Can it be modelled as damped or undamped? Free or forced? If forced, is the excitation due to an applied force or to ground motion? Once you have done this, you will know which set of equations will be applicable.

The parameters you will need to know, or calculate, will be the mass, m , the stiffness, k , and, if there is damping, then either the damping coefficient, c , or the damping ratio, ζ . The natural frequency ω is always of interest, and you will need to calculate that. If the vibration is forced,

then the excitation frequency Ω will need to be known, and, hence, the frequency ratio, r can be determined. Armed with these parameters, the response of the system (how it behaves in terms of its displacement from the equilibrium position) can be resolved.

Measurement of vibration

An accelerometer is a piezo-electric device that is frequently used to monitor vibration. It is attached rigidly to the system of interest, such as a wing of an aircraft under test. It measures the acceleration at the point where it is attached, giving a voltage output that is directly proportional to the acceleration. This voltage can be integrated by suitable electronic instrumentation to provide values of velocity and displacement. The use of an amplifier and an oscilloscope or a frequency meter allows the frequency of vibration to be determined. To determine the undamped natural frequency, the object of interest, an engine perhaps, should be supported as freely as possible, because any sources of friction will introduce damping. If it is then given a light tap, or a small initial displacement, and released it will vibrate at its natural frequency, which can be determined from the accelerometer output. Ideally, the accelerometer should have a small mass compared with the object under test, although the extra mass can be taken into account.