

Section 3. Rotating Unbalance

We considered forced vibration due to the application of a sinusoidal force in Section 2. A common source of such a sinusoidal force is unbalance in a rotating machine or rotor. You may have experienced the effect if you have ever driven a car where the wheels are not balanced; you will have noticed that at a particular speed, the car will shake, sometimes quite violently. At this speed, the rotational speed of the wheels is such as to be close to the natural frequency of the car on its suspension, so that the amplitude becomes a maximum. In this section we will look at how we can describe such a phenomenon more precisely, in a mathematical way.

Rotating machines include turbines, electric motors and electric generators, as well as fans, or rotating shafts. Rotating shafts experience a special kind of response, known as “whirl”, but we will not be able to consider them in this course.

Let us suppose that a rotating machine of mass m , can be modelled as being mounted on a spring of stiffness, k , to a fixed support, and that there is viscous damping in the system, with a damping coefficient, c . We will also suppose that the machine is constrained to move vertically, so that this is a single degree-of-freedom system. The rotor is unbalanced if its centre of mass does not coincide with the centre of rotation. Suppose that the unbalance can be represented by a mass m_u at a distance e from the centre of rotation. (e is sometimes called the eccentricity.) Let the angular speed of rotation of the rotor be Ω . The system is illustrated in Figure 3.1.

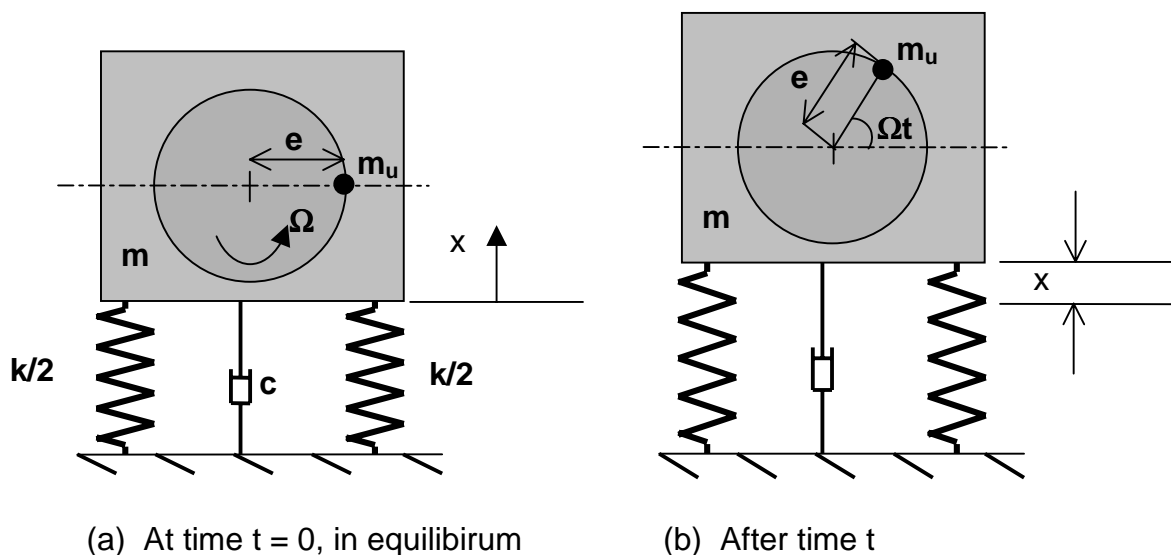


Figure 3.1. Model of an unbalanced rotating machine

After a time t , the out-of-balance mass will have moved through an angular displacement Ωt . If x is the displacement of the mass ($m - m_u$) from the equilibrium position after a time t , the displacement of the out-of-balance mass, m_u , will be $(x + e \sin \Omega t)$. The forces acting on the combined mass are

$-kx$ and $-c\dot{x}$ (i.e. they act in the opposite direction to the positive x -direction). The equation of motion is, therefore:

$$-kx - c\dot{x} = (m - m_u) \frac{d^2 x}{dt^2} + m_u \frac{d^2}{dt^2} (x + e \sin \Omega t)$$

$$-kx - c\dot{x} = m\ddot{x} - m_u\ddot{x} + m_u\ddot{x} + m_u(-\Omega^2 e \sin \Omega t)$$

This can be written as $m\ddot{x} + c\dot{x} + kx = m_u\Omega^2 e \sin \Omega t$

Now this is of the same form as the equation of motion for damped forced vibration that you met in the previous section:

$$m\ddot{x} + c\dot{x} + kx = F_0 \sin \Omega t$$

but F_0 has been replaced by $m_u\Omega^2 e$. So the excitation is due to an applied force, $F_0 \sin \Omega t$, where $F_0 = m_u e \Omega^2$. This is the centripetal force of the mass m_u towards the centre of rotation. If the rotor is balanced, e is 0, so this force does not exist. If e is only small the amplitude of the exciting force is also small. The amplitude of the excitation depends on both the out-of-balance mass, and its effective distance from the axis of rotation, as well as on the rotational speed.

The response will be given by $x = X_0 \sin(\Omega t - \phi)$ where

$$X_0 = \frac{\omega^2 m_u \Omega^2 e / k}{\sqrt{(\omega^2 - \Omega^2)^2 + (2\zeta\omega\Omega)^2}} = \frac{m_u \Omega^2 e / k}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} = \frac{m_u r^2 e / m}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} \quad (3.1)$$

and
$$\tan \phi = \frac{2\zeta\omega\Omega}{\omega^2 - \Omega^2} = \frac{2\zeta r}{1 - r^2} \quad (3.2)$$

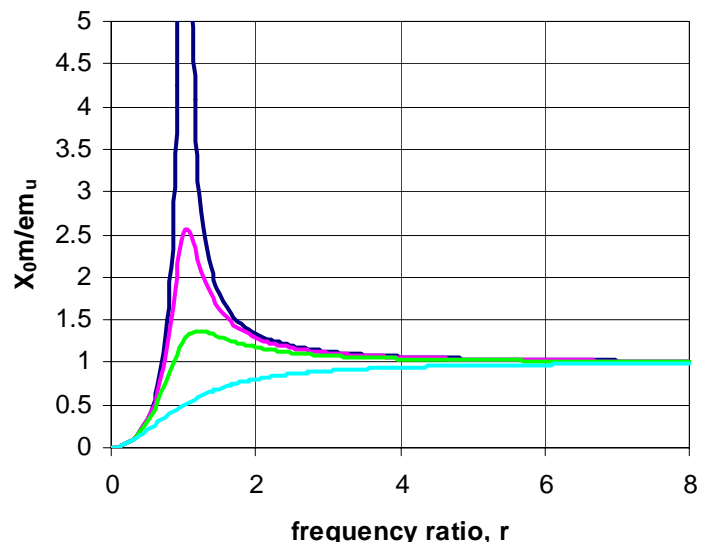
and, as before, ω is the natural frequency, $\sqrt{\frac{k}{m}}$, $\zeta = \frac{c}{2\sqrt{km}}$ and $r = \frac{\Omega}{\omega}$

If we write equation (3.1) in a non-dimensional form it is easier to see how in general the response depends on the frequency ratio, r , and on the damping ratio ζ . Re-writing equation (3.1), therefore:

$$\frac{X_0}{e} \frac{m}{m_u} = \frac{r^2}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}$$

Figure 3.2 is obtained by plotting $\frac{X_0 m}{e m_u}$ against r .

Figure 3.2. The response of an unbalanced rotor as a function of the frequency



This shows that at low speeds, where r is small, the amplitude of the motion of the mass ($m - m_u$) is nearly 0, while at very large values of r , the amplitude becomes constant, at a value equal to em_u/m , and it doesn't matter what the damping in the system is at all, nor does the actual speed then matter. This explains why, if you are bold enough to speed up when your car with unbalanced wheels begins to shake, the shaking stops, as the value of r becomes much greater than 1.

The maximum amplitude occurs when $r = \sqrt{1 - 2\zeta^2}$

At resonance, when $r = 1$, the amplitude of the mass ($m - m_u$) is $X_0 = \frac{m_u e}{2\zeta m}$ so

its magnitude is reduced if there is damping in the system.

At resonance, the phase angle $\phi = 90^\circ$ and the response lags the excitation by 90° . This means that when the mass ($m - m_u$) is moving upwards through its position of equilibrium the unbalance mass m_u is directly above the centre of rotation.

A rotor can be balanced by placing a balancing mass, m_b , on the rotor diametrically opposite m_u , and at a distance, h , such that $m_b h = m_u e$.

This mass m_b then produces an excitation force that is exactly equal and opposite to that produced by the out-of-balance, so that there is no resultant excitation force.

Example

A 40 kg motor is similar to the system shown in Figure 3.1. It is supported by *four* springs each of stiffness 250 Nm^{-1} . The rotor is unbalanced such that the unbalance effect is equivalent to a mass of 5 kg located 50 mm from the axis of rotation. Find the amplitude of vibration and the force transmitted to the foundation when the speed of the motor is (a) $1000 \text{ rev min}^{-1}$, (b) 60 rev min^{-1} . The damping ratio, $\zeta = 0.15$.

Solution:

Write down what you know: $m = 40 \text{ kg}$ $k = 4 \cdot 250 = 1000 \text{ Nm}^{-1}$

$m_u = 5 \text{ kg}$ $e = 50 \text{ mm} = 0.05 \text{ m}$ $\zeta = 0.15$

Speed of motor = $1000 \text{ rev min}^{-1}$, $\therefore \Omega = \frac{1000 \cdot 2\pi}{60} = 104.7 \text{ rad s}^{-1}$

First find the natural frequency and r .

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{1000}{40}} = 5 \text{ rad s}^{-1} \qquad \therefore r = \frac{\Omega}{\omega} = \frac{104.7}{5} = 20.94$$

From equation (3.1),

$$X_0 = \frac{m_u r^2 e / m}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = \frac{5 * 20.94^2 * 0.05 / 40}{\sqrt{(1-20.94^2)^2 + (2 * 0.15 * 20.94)^2}}$$

$$= \frac{2.74}{\sqrt{191391.9 + 39.5}} = \frac{2.74}{437.5} = 6.3 * 10^{-3} m$$

So the amplitude of the vibration of the motor at 1000 rev min⁻¹ is 6.3 mm.

$$\text{From section 2, } P_0 = F_0 \left[\frac{1 + (2\zeta r)^2}{(1-r^2)^2 + (2\zeta r)^2} \right]^{1/2} = m_u e \Omega^2 \left[\frac{1 + (2\zeta r)^2}{(1-r^2)^2 + (2\zeta r)^2} \right]^{1/2}$$

$$\therefore P_0 = 5 * 0.05 * 104.7^2 \left[\frac{1 + (2 * 0.15 * 20.94)^2}{(1-20.94^2)^2 + (2 * 0.15 * 20.94)^2} \right]^{1/2} = 2740.5 \sqrt{\frac{40.46}{191431.4}} = 39.84$$

The amplitude of the force transmitted is 40 N.

$$(b) \Omega = \frac{60 * 2\pi}{60} = 6.28 \text{ rad s}^{-1} \text{ and } \therefore r = \frac{6.28}{5} = 1.26$$

$$X_0 = \frac{m_u r^2 e / m}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = \frac{5 * 1.26^2 * 0.05 / 40}{\sqrt{(1-1.26^2)^2 + (2 * 0.15 * 1.26)^2}}$$

$$= \frac{0.00992}{\sqrt{0.345 + 0.143}} = \frac{0.00992}{0.698} = 0.014 m$$

$$\therefore P_0 = 5 * 0.05 * 6.28^2 \left[\frac{1 + (2 * 0.15 * 1.26)^2}{(1-1.26^2)^2 + (2 * 0.15 * 1.26)^2} \right]^{1/2} = 9.86 \sqrt{\frac{1.143}{0.488}} = 15.1$$

The amplitude of the response is 14 mm and the force transmitted is 15 N.

In this case, the displacement amplitude is more than twice its value at the higher speed, which means more than 4 times the peak-to-peak displacement. The force transmitted to the foundations however has reduced. The more damping there is, the less critical is resonant frequency as far as the force transmitted is concerned. With a damping ratio of more than 0.1 the force transmitted is increased when the speed of rotation increases.

This example illustrates the problems that can arise during machine start-up. Because as the speed gradually increases to the operating speed it will go through a point when r is 1, and the amplitude of vibration can become momentarily large. If there is only a small amount of damping in the system then the force transmitted can also become significant.