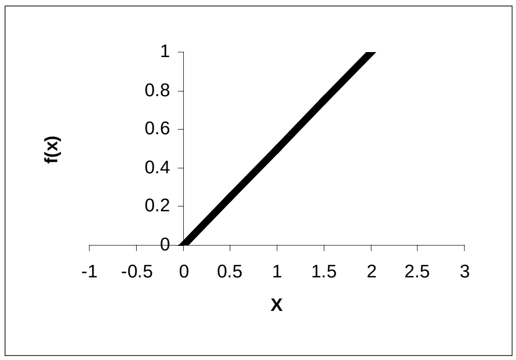
Outline Solutions to Tutorial Sheet 6

1. (i) For a pdf we require $\int_{-\infty}^{\infty} f(x)dx = 1$

   Thus $\int_0^2 kxdx = 1 \rightarrow \left[k\frac{x^2}{2} \right]_0^2 = 1 \rightarrow k\left(\frac{4}{2}\right) - k(0) = 1 \rightarrow$ k = 0.5

   Sketch



   (ii) $E(X) = \int_{-\infty}^{\infty} x.f(x)dx = \int_0^2 x.0.5xdx = 0.5\left[\frac{x^3}{3}\right]_0^2 = 1.333$

   (iii) $F(x) = \int_{-\infty}^{x} f(x)dx = 0.5\int_0^x xdx = 0.5\frac{x^2}{2}$

   Thus the cdf is $F(x) = \begin{cases} 0 & x < 0 \\ \frac{x^2}{4} & 0 < x < 2 \\ 1 & x > 2 \end{cases}$

   The median is the value *m* such that $p(x < m) = F(m) = \frac{1}{2}$

   Thus we need *m* such that $\frac{m^2}{4} = \frac{1}{2}$. Hence the median *m* is $\sqrt{2} = 1.414$

   (iv) Note that the mean (1.333) is less than the median (1.414). This is to be expected since the shape of the distribution is *negatively skew*.

2. $E(X) = \int_{-\infty}^{\infty} x.f(x)dx = \int_{1}^{\infty} x\left(\frac{2}{x^3}\right)dx = 2\int_{1}^{\infty} x^{-2}dx = 2\left[-\frac{1}{x}\right]_1^{\infty} = (0) - 2(-1) = 2$ microns

$$p(x < 1.1) = \int_{1}^{1.1} \frac{2}{x^3}dx = 2\left[\frac{x^{-2}}{-2}\right]_1^{1.1} = \left[\frac{1}{x^2}\right]_{1.1}^{1} = 1 - \frac{1}{1.1^2} = 1 - 0.826 = 0.174$$

Thus 17.4% of particles will get through the filter.

3. (i) $p(X > 0.5) = 1 - p(X < 0.5) = 1 - \int_{0}^{0.5} x dx = 0.875$

(ii) $p(X < 1.5) = \int_{0}^{1} x dx + \int_{1}^{1.5} (2 - x)dx = 0.875$

(iii) $p(0.5 < X < 1.5) = p(X < 1.5) - p(X < 0.5) = 0.875 - (1 - 0.875) = 0.75$

(iv) $F(x) = \begin{cases} 0 & x < 0 \\ x^2/2 & 0 \le x \le 1 \\ -(x^2 - 4x + 2)/2 & 1 \le x \le 2 \\ 1 & \text{otherwise;} \end{cases}$

(v) $E(X) = \int_{-\infty}^{\infty} x.f(x)dx = \int_{0}^{1} x.x dx + \int_{1}^{2} x.(2 - x)dx = \frac{1}{3} + \frac{2}{3} = 1$

A sketch of the pdf shows it to be a triangular distribution **symmetrical** about X=1. Hence the mean (and the median and the mode) will be 1.

4. (i) Since $F(x) = p(X < x) = 1\text{-exp}(-x^2/16)$ it follows that $p(X>x) = \exp(-x^2/16)$
Hence $p(X>4) = \exp(-16/16) = e^{-1} = 0.368$

(ii) We require the height of the wall, h, such that $p(X>h) = 0.02$

Hence, we require $e^{-\frac{h^2}{16}} = 0.02$

Solving this gives $\frac{h^2}{16} = -\log_e(0.02) = 3.9120$

So the required height of the wall is $\sqrt{16 \times 3.9120} = 7.912$ meters.