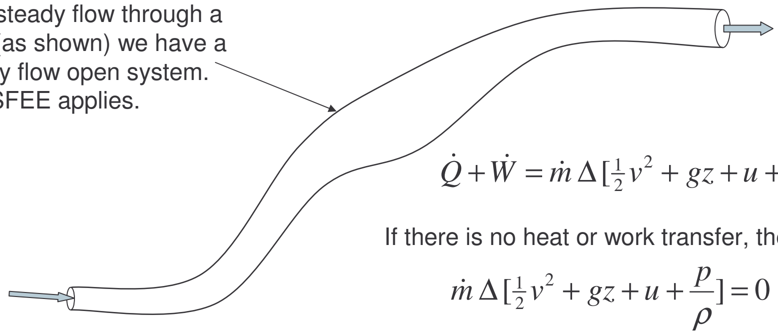
ENERGY

With steady flow through a tube (as shown) we have a steady flow open system. The SFEE applies.



$$\dot{Q} + \dot{W} = \dot{m}\,\Delta[\tfrac{1}{2}v^2 + gz + u + \frac{p}{\rho}]$$

If there is no heat or work transfer, then:

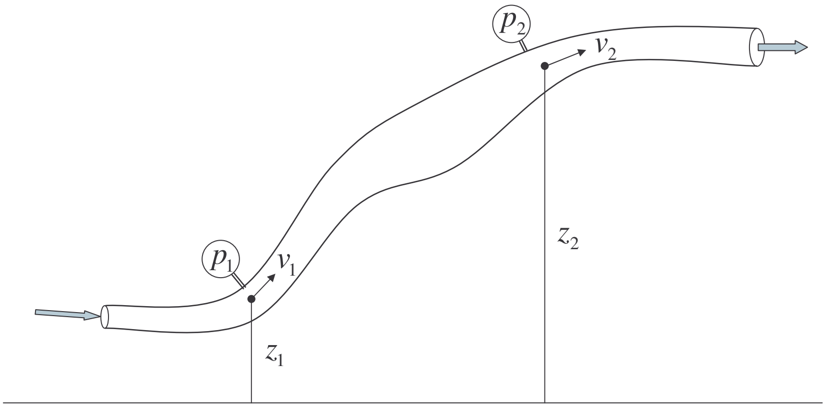
$$\dot{m}\,\Delta[\tfrac{1}{2}v^2 + gz + u + \frac{p}{\rho}] = 0$$

$$or\ \tfrac{1}{2}v^2 + gz + u + \frac{p}{\rho} = const$$

If the flow is incompressible and frictionless there will be no frictionally generated heat and therefore the specific internal energy $\boldsymbol{u}$, will remain constant:

$$\therefore \quad \boxed{\tfrac{1}{2}v^2 + gz + \frac{p}{\rho} = const}$$

This equation is known as Bernoulli's equation. It applies to incompressible fluid flow over relatively short distances where fluid frictional effects are minimal.



$$\tfrac{1}{2}{v_1}^2 + gz_1 + \frac{p_1}{\rho} = \tfrac{1}{2}{v_2}^2 + gz_2 + \frac{p_2}{\rho}$$

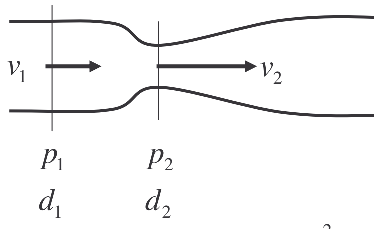
no friction between (1) and (2)

Note that each term of the equation is energy/unit mass (specific energy).

If energy **<u>is</u>** lost from (1) to (2) then the LHS > RHS

(we'll look in greater detail at this later)

Design Brief: Design a flow meter which uses the principle of Bernoulli's equation.



From continuity: $\frac{\pi}{4} d_1^2 v_1 = \frac{\pi}{4} d_2^2 v_2$

$$\therefore \quad v_2 = \left\{\frac{d_1}{d_2}\right\}^2 v_1$$

From Bernoulli: $\frac{1}{2} v_1^2 + gz_1 + \frac{p_1}{\rho} = \frac{1}{2} v_2^2 + gz_2 + \frac{p_2}{\rho}$

$$\frac{1}{2} v_1^2 + \frac{p_1}{\rho} = \frac{1}{2}\left\{\frac{d_1}{d_2}\right\}^4 v_1^2 + \frac{p_2}{\rho}$$

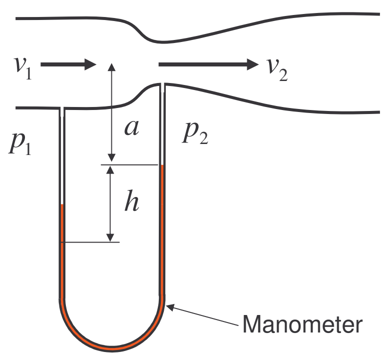
$$\frac{1}{2} v_1^2 - \frac{1}{2}\left\{\frac{d_1}{d_2}\right\}^4 v_1^2 = \frac{p_2}{\rho} - \frac{p_1}{\rho}$$

$$v_1^2 = \frac{\frac{2p_2}{\rho} - \frac{2p_1}{\rho}}{1 - \left\{\frac{d_1}{d_2}\right\}^4} \qquad v_1 = \sqrt{\frac{2(p_2 - p_1)}{\rho\left(1 - \left\{\frac{d_1}{d_2}\right\}^4\right)}}$$

Venturi meter

$$\dot{m} = \rho \frac{\pi}{4} d_1^2 v_1$$

If the pressure difference is measured using a manometer:



$$v_1 = \sqrt{\frac{2(p_1 - p_2)}{\rho\left(\left\{\frac{d_1}{d_2}\right\}^4 - 1\right)}}$$

$$\dot{m} = \rho \frac{\pi}{4} d_1^2 v_1$$

For the manometer:

$$p_1 + \rho g(a + h) = p_2 + \rho g a + \rho_M g h$$

$$p_1 - p_2 = (\rho_M - \rho) g h$$

$$\frac{p_1 - p_2}{\rho} = \left(\frac{\rho_M}{\rho} - 1\right) g h$$

$$v_1 = \sqrt{\frac{2\left(\frac{\rho_M}{\rho} - 1\right) g h}{\left\{\frac{d_1}{d_2}\right\}^4 - 1}}$$

$$\dot{m} = \rho \frac{\pi}{4} d_1^2 \sqrt{\frac{2\left(\frac{\rho_M}{\rho} - 1\right) g h}{\left\{\frac{d_1}{d_2}\right\}^4 - 1}}$$

$$\dot{m} = \rho \frac{\pi}{4} d_1^{\,2} \sqrt{\frac{2(\frac{\rho_M}{\rho} - 1)gh}{\left\{\frac{d_1}{d_2}\right\}^4 - 1}}$$

In practice the operation of a Venturi meter is not absolutely frictionless. To take account of frictional effects a coefficient is introduced known as the discharge coefficient: $C_{D.}$
The value of this coefficient depends on the design of the flow meter, and upon the exact conditions of flow.
Typical values are 0.95 for a Venturi meter, and 0.6 for an orifice meter.
(An orifice meter is simply a hole cut in a flat plate).
The values for C_D can be looked up in British Standards (e.g. BS 1042)

$$\dot{m} = C_D \rho \frac{\pi}{4} d_1^{\,2} \sqrt{\frac{2(\frac{\rho_M}{\rho} - 1)gh}{\left\{\frac{d_1}{d_2}\right\}^4 - 1}}$$

Forms of Bernoulli's Equation (1)

Each term of Bernoulli's equation written in this form: $\tfrac{1}{2}v^2 + gz + \dfrac{p}{\rho} = const$

is energy/unit mass (specific energy) as we noted above..

However, since the equation applies to incompressible fluids the density of the fluid remains constant. We can therefore multiply each term by density to obtain:

$$\tfrac{1}{2}\rho v^2 + \rho g z + p = const$$

In this form, each term is that of pressure. Each term can be identified:

$$\tfrac{1}{2}\rho v^2 + \rho g z + p = const$$

dynamic pressure — by virtue of the fluid's velocity

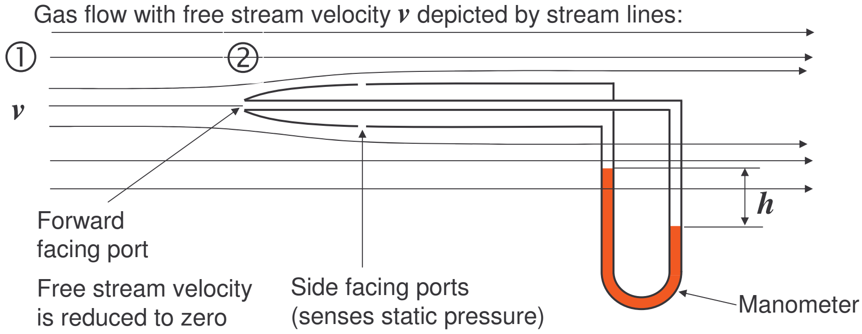
gravitational pressure — by virtue of gravity

static pressure — sensed by a particle travelling with the flow

This form of the equation is used in fluid velocity measurement with a pitot-static tube.

Example: Measure the free stream velocity of a gas flow using a pitot-static tube:

Gas flow with free stream velocity $\boldsymbol{v}$ depicted by stream lines:



Applying Bernoulli's equation between (1) & (2):

$$\tfrac{1}{2}\rho v^2 + \rho g z + p_1 = \tfrac{1}{2}\rho 0^2 + \rho g z + p_2$$

$$\tfrac{1}{2}\rho v^2 = p_2 - p_1$$

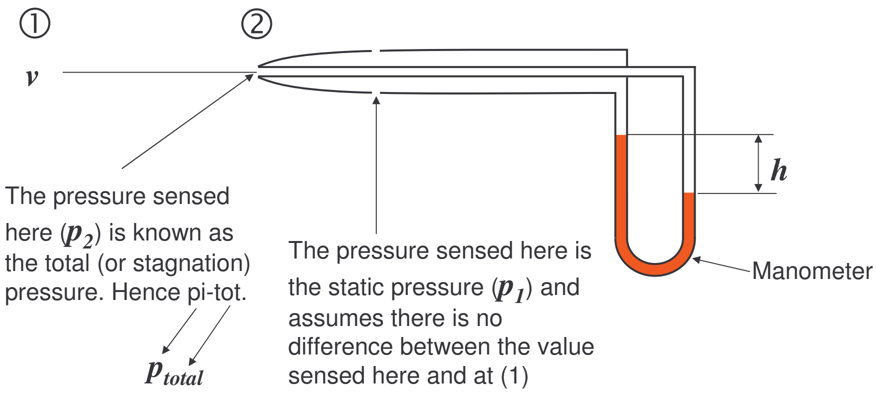
For the manometer: (as before)

$$\frac{p_2 - p_1}{\rho} = \left(\frac{\rho_M}{\rho} - 1\right) g h$$

$$v = \sqrt{\frac{2(p_2 - p_1)}{\rho}}$$

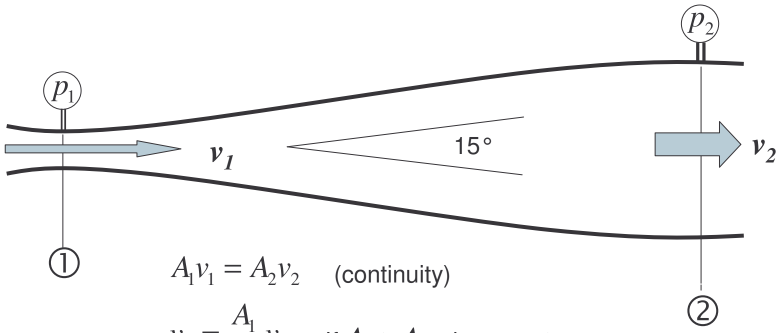
$$\text{If} \quad \rho_M >> \rho \quad \text{then: } p_2 - p_1 = \rho_M g h$$

$$v = \sqrt{\frac{2\rho_M g h}{\rho}}$$



$$p_{total} = p_{static} + \tfrac{1}{2}\rho v^2$$

Example: Conversion of a high speed gas stream to a low speed gas stream:



$$A_1 v_1 = A_2 v_2 \quad \text{(continuity)}$$

$$v_2 = \frac{A_1}{A_2} v_1 \quad \text{If } A_2 > A_1 \text{ then } v_2 < v_1$$

What happens to p_2? Apply Bernoulli: $\frac{1}{2}\rho v_1^2 + p_1 = \frac{1}{2}\rho v_2^2 + p_2$

$$p_2 = p_1 + \tfrac{1}{2}\rho(v_1^2 - v_2^2)$$

$$\text{If } v_2 \approx 0 \text{ then}$$

$$p_2 = p_1 + \tfrac{1}{2}\rho v_1^2$$

This device is known as a **diffuser**. The angle of diffusion has an upper limit of ~ 15° because the flow is against the static pressure gradient.

Forms of Bernoulli's Equation (2)

Each term of Bernoulli's equation written in this form: $\frac{1}{2}v^2 + gz + \frac{p}{\rho} = const$

is energy/unit mass (specific energy) as we noted above..

However, g is also constant so by dividing each term by g we can write:

$$\frac{v^2}{2g} + z + \frac{p}{\rho g} = const$$

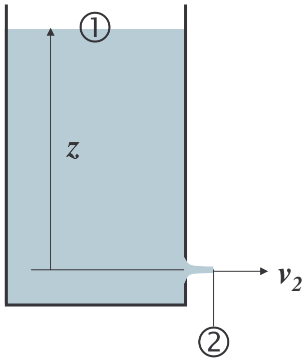
In this form, each term is that of height (or **head**). Each term can be identified:

$$\frac{v^2}{2g} + z + \frac{p}{\rho g} = const$$

- dynamic head ($\frac{v^2}{2g}$): by virtue of the fluid's velocity
- gravitational head (z): the actual height above a datum
- static head ($\frac{p}{\rho g}$): the height to which a piezometer would rise

This form of the equation is often used when flows occur between different fluid levels, either because of being pumped or flowing by gravity.

Example: What is the velocity of a fluid jet emerging from a hole in the side of tank?



$$\frac{v^2}{2g} + z + \frac{p}{\rho g} = const$$

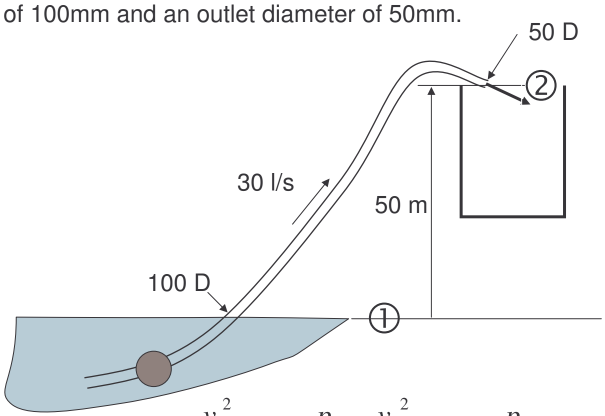
$$\frac{v_1^2}{2g} + z + \frac{p_1}{\rho g} = \frac{v_2^2}{2g} + 0 + \frac{p_2}{\rho g}$$

$v_1 \approx 0$ and $p_1 = p_2$ so

$$\frac{v_2^2}{2g} = z \quad \text{or} \quad v_2 = \sqrt{2gz}$$

As the tank empties z decreases. Since we have v as a function of z for a given hole size we can work out how long it would take the tank to empty.

Example: Water is pumped from a lake to a holding tank requiring a lift of 50m. What pressure head is required at lake level to obtain a flow rate of 30 l/s with a pipe diameter of 100mm and an outlet diameter of 50mm.



from continuity:

$$\dot{V} = A_1 v_1 = A_2 v_2$$

$$30\times10^{-3} = \frac{\pi}{4}(100\times10^{-3})^2 v_1$$

$$30\times10^{-3} = \frac{\pi}{4}(50\times10^{-3})^2 v_2$$

$v_1 = 3.82\,\text{m/s}$ and

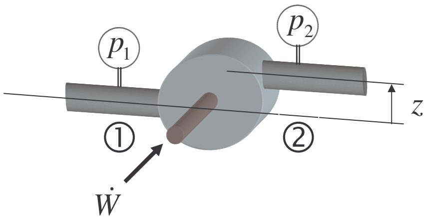
$v_2 = 15.28\,\text{m/s}$

Bernoulli: $$\frac{v_1^2}{2g} + 0 + \frac{p_1}{\rho g} = \frac{v_2^2}{2g} + 50 + \frac{p_2}{\rho g} \quad \text{where} \quad p_2 = 101.325\,kPa$$

$$\frac{p_1}{\rho g} = \frac{v_2^2}{2g} - \frac{v_1^2}{2g} + 50 + \frac{p_2}{\rho g} = \frac{15.28^2}{2\times9.81} - \frac{3.82^2}{2\times9.81} + 50 + \frac{101325}{1000\times9.81}$$

$$\frac{p_1}{\rho g} = 61.16\,m$$

PUMPS



$$\dot{Q} + \dot{W} = \dot{m}\,\Delta[\tfrac{1}{2}v^2 + gz + u + \frac{p}{\rho}]$$

$$\dot{W} = \dot{m}[\tfrac{1}{2}({v_2}^2 - {v_1}^2) + g(z_2 - z_1) + (\frac{p_2}{\rho} - \frac{p_1}{\rho})]$$

if $v_1 = v_2$

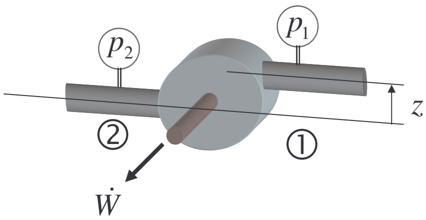
$$\dot{W} = \dot{m}[gz + (\frac{p_2}{\rho} - \frac{p_1}{\rho})]$$

or $$\frac{\dot{W}}{\dot{m}g} = z + (\frac{p_2}{\rho g} - \frac{p_1}{\rho g})$$

In practice some of the actual power input is lost to friction:

$$\dot{W}_{actual} = \frac{\dot{W}}{\eta_{pump}}$$

TURBINES



$$\dot{Q} + \dot{W} = \dot{m}\,\Delta[\tfrac{1}{2}v^2 + gz + u + \frac{p}{\rho}]$$

$$\dot{W} = \dot{m}[\tfrac{1}{2}({v_2}^2 - {v_1}^2) + g(z_2 - z_1) + (\frac{p_2}{\rho} - \frac{p_1}{\rho})]$$

if $v_1 = v_2$

$$\dot{W} = \dot{m}[gz + (\frac{p_2}{\rho} - \frac{p_1}{\rho})]$$

or $$\frac{\dot{W}}{\dot{m}g} = z + (\frac{p_2}{\rho g} - \frac{p_1}{\rho g})$$

In practice some of the actual power output is lost to friction:

$$\dot{W}_{actual} = \dot{W}\eta_{turbine}$$