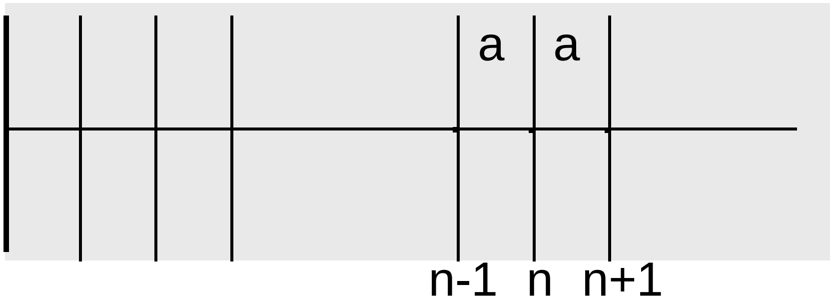
1-D TRANSIENT CONDUCTION

The equation to be solved is:-

$$\frac{\partial T}{\partial t} = \frac{\lambda}{\rho c_p}\left(\frac{\partial^2 T}{\partial x^2} + \cancel{\frac{\partial^2 T}{\partial y^2}} + \cancel{\frac{\partial^2 T}{\partial z^2}}\right) + \frac{\dot{Q}'''}{\rho c_p}$$

$$\frac{\partial T}{\partial t} = \frac{\lambda}{\rho c_p}\left(\frac{\partial^2 T}{\partial x^2}\right) + \cancel{\frac{\dot{Q}'''}{\rho c_p}}$$

We'll assume initially that there is no internal heat generation, and divide the 1-D system into equi-spaced planes.



At layer n the solution may be written:-

$$\frac{T_{n,1} - T_{n,0}}{\Delta t} = \left(\frac{\dfrac{T_{n-1} - T_n}{a} - \dfrac{T_n - T_{n+1}}{a}}{a}\right)\frac{\lambda}{\rho C_P} \qquad \text{where} \quad \alpha = \frac{\lambda}{\rho C_P}$$

This expression can be used to find the temperature at plane 'n' at time instant '1' from the conditions existing at time instant '0'.

Re-arranging gives:-

$$T_{n,1} = T_{n,0} + \left\{T_{n-1,0} - 2T_{n,0} + T_{n+1,0}\right\}\frac{\alpha \Delta t}{a^2}$$

Let $\frac{\alpha \Delta t}{a^2} = F$ (the non-dimensional grid Fourier Number)

then
$$\boxed{T_{n,1} = F\left\{T_{n-1,0} + \left(2 - \frac{1}{F}\right)T_{n,0} + T_{n+1,0}\right\}}$$

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The only unknown in this equation is $T_{n,1}$ the temperature at layer 'n' after time interval Δt.

Thus, from a knowledge of the conditions at time zero, temperatures in successive layers can be found after each time interval.

The equation gives an EXPLICIT solution for each temperature in turn and lends itself to tabular, graphical or computer solution.

Examination shows that if the coefficient of $T_{n,0}$ is negative
($F < 0.5$) the solution will become oscillatory and eventually unstable.

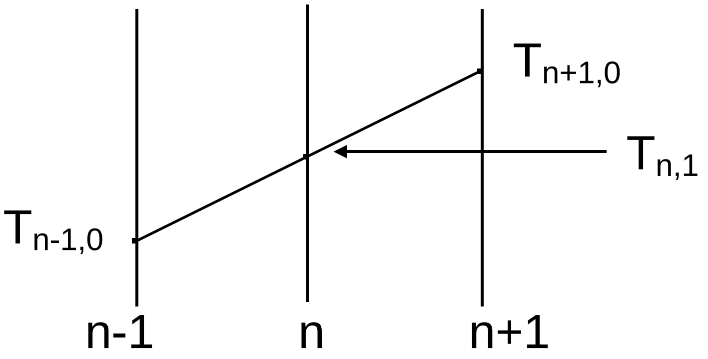
Schmidt Graphical Solution

If F= 0.5 then the equation becomes:-

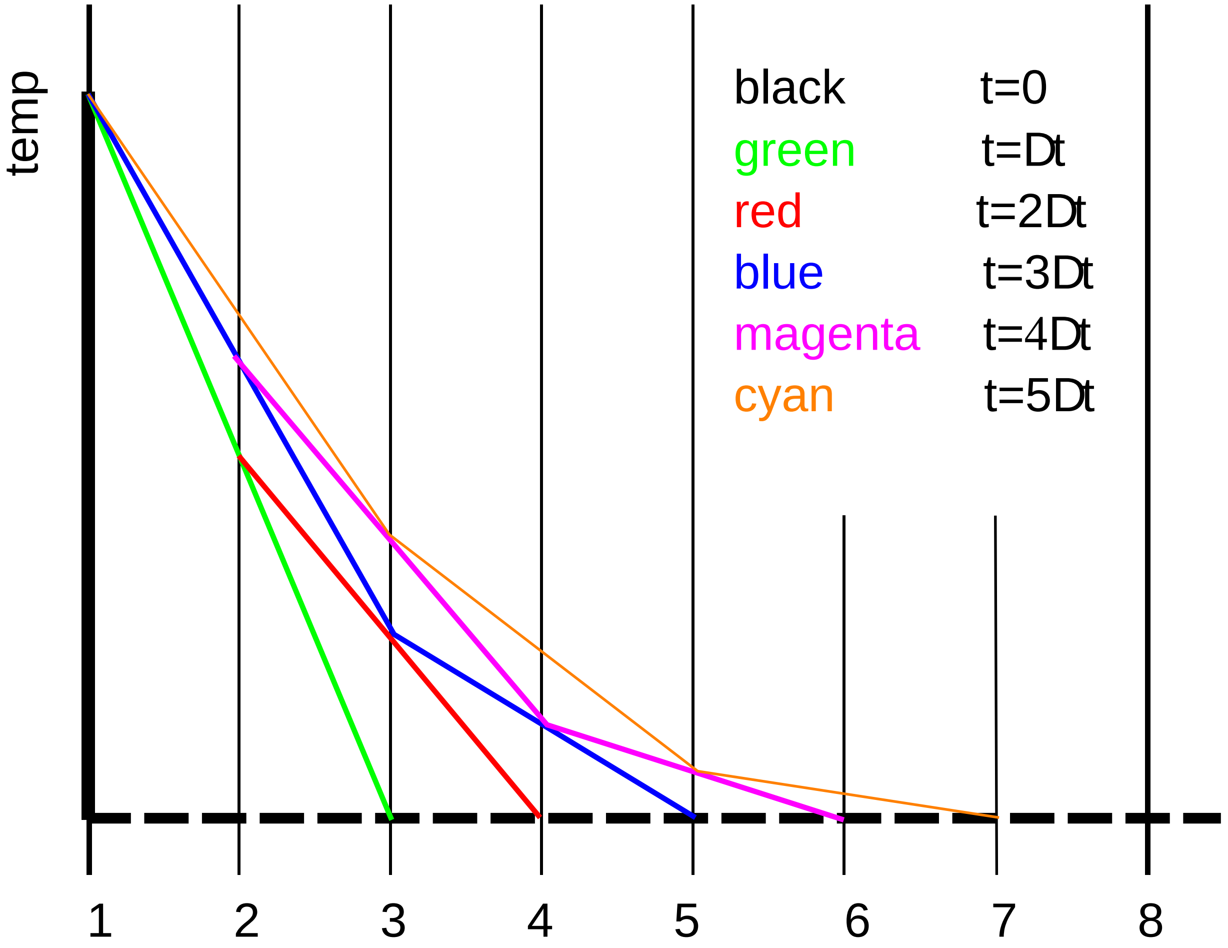
$$T_{n,1} = \frac{T_{n-1,0} + T_{n+1,0}}{2}$$

This can be solved using graphical construction

Every other plane can be updated at each time instant.



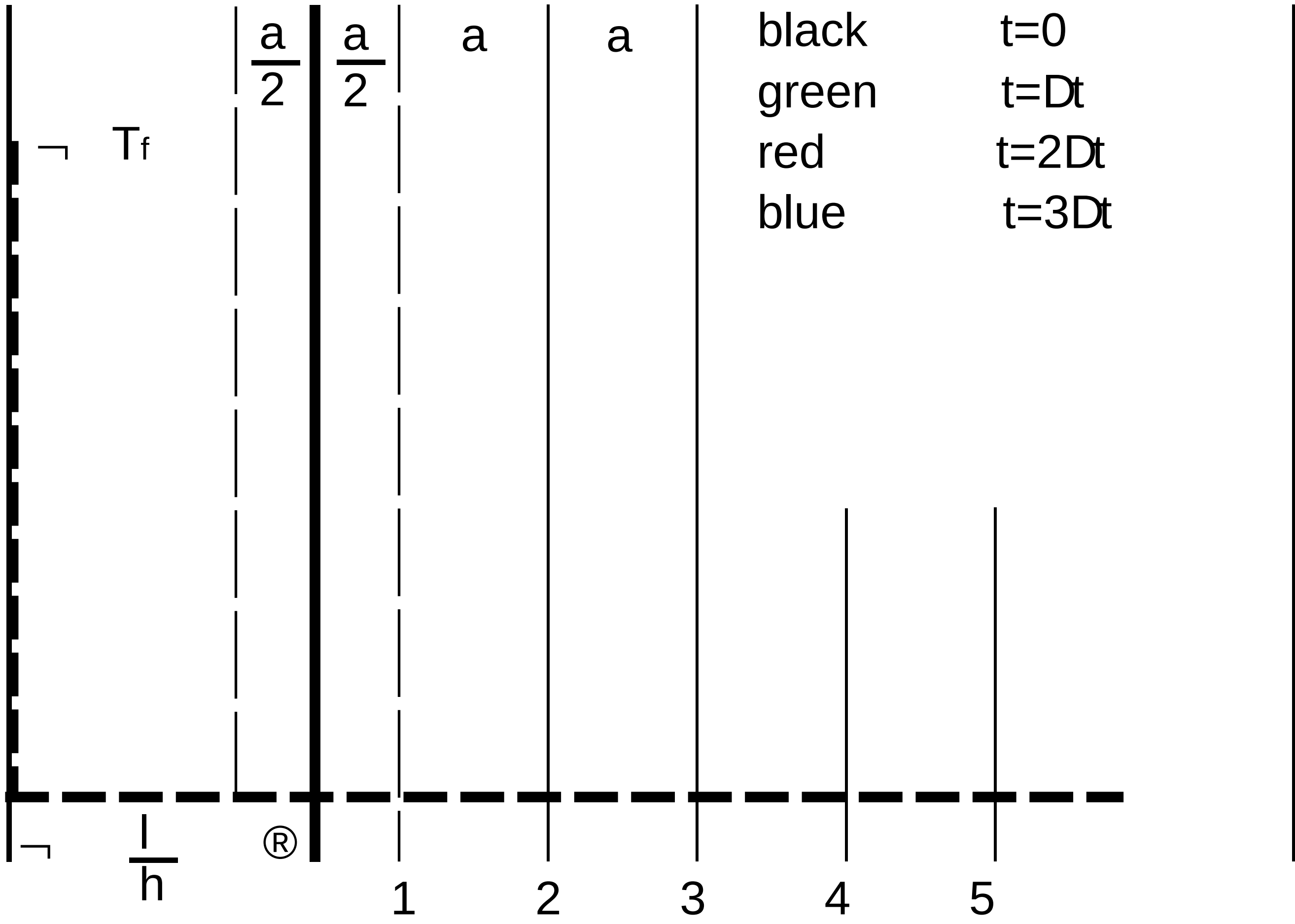
Example : wall at uniform temperature where one surface is suddenly raised to a higher temperature.



A convective boundary can be incorporated into the Schmidt method by equating the convection and conduction at the boundary at time zero. ie

or

ie, an extra 'conductive' layer of thickness l /h is drawn outside the wall and used with a false half-layer within the



HEAT TRANSFER into the CONDUCTOR

The total heat transfer into the conductor may be found either by summing wrt time or distance (layer). ie

or

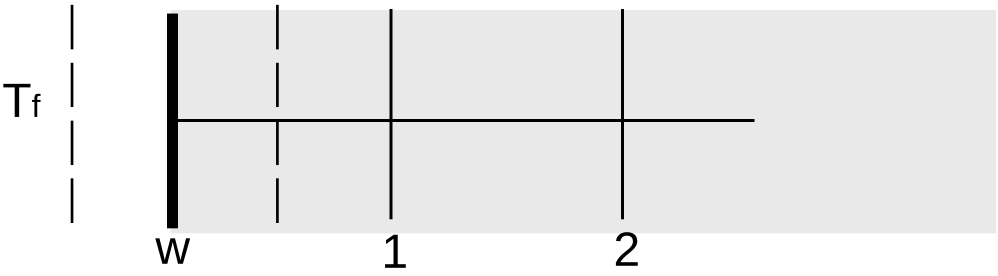
$$\boxed{Q = \sum^{\text{all layers}} m\, C_p \left(T_{\text{final}} - T_{\text{zero}}\right)}$$

CONVECTIVE BOUNDARIES - General

If we go back to our original equation :

$$T_{n,1} = F\left\{T_{n-1,0} + \left(2 - \frac{1}{F}\right)T_{n,0} + T_{n+1,0}\right\}$$

we can modify it to take account of a convective boundary condition by energy balance at the surface 'half-layer'.



For the surface half-layer:

Convection IN/OUT - Conduction OUT/IN = Stored Energy

As before $\frac{\alpha \Delta t}{a^2} = F$ (the grid Fourier No.)

and put $\frac{ha}{\lambda} = Bi$ (the grid Biot No.)

hence:

If the coefficient of $T_{w,0}$ is negative ($F + F\,Bi > 0.5$) the solution will become oscillatory and eventually unstable.

NB the expression remains <u>explicit;</u> if we know the conditions at time t, we can find conditions at time, t + Δt.

Alternative Finite Difference methods

Because of the instability of the explicit equations they often call for excessive calculations. This is especially so in 2-D or 3-D where the stability criteria becomes more constraining. e.g. in 2-D F must be £ 0.25.

The equation:-

is based on 'backward' time differences. ie We found conditions at plane 'n' at time instant '1' in terms of the conditions which existed at planes n-1, n, & n+1 at time instant '0'.

We could equally well have based the equation on 'forward' time differences. ie We could find conditions at plane 'n' at time instant '1' in terms of the conditions which will exist at the planes n-1, n, & n+1 at time instant '1'.

If we do this, the equation becomes:-

The obvious snag is that we don't <u>know</u> conditions at time instant '1', so therefore we have to write equations for every plane for a given time and solve them simultaneously.

Alternatively, we may solve them iteratively!

Writing the equation using Fourier No. (as before) gives:-

We can solve the temperatures by writing the equation in 'residual' form (= R) , and iterating to reduce the residuals to zero.

An improved value for $T_{n,1}$ is given by:-

This method of solution is an 'implicit' technique because we cannot solve for solutions at each point directly, as we did using the previous explicit method.

It is found that this method is completely stable and we may use any value of F which gives the required accuracy.

We may also write the convective boundary condition in implicit form.

It is found that compared with the exact solution the explicit method tends to underestimate the answer, whereas the implicit method tends to overestimate the answer.

A method using the average of the two above methods (due to Crank & Nicolson) gives almost correct answers.

(See Bacon, Basic Heat Transfer p.)

Exact Methods

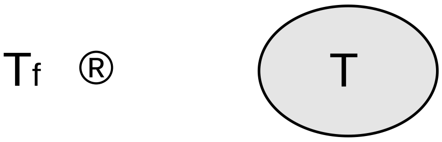
For some regular solids (eg slabs, spheres, cylinders etc.) exact solutions are available in the form of non-dimensional charts.

It is also possible to solve some 3-D solutions by 'decomposition' using 1-D solutions.

LUMPED CAPACITY SYSTEMS

If a solid has a high thermal conductivity, and it loses heat comparatively slowly, the temperature variations within it are small and the whole solid can be assumed to be at a constant temperature throughout.

ie l $_{\text{solid}}$ >> h $_{\text{fluid}}$ so that $\frac{ha}{l}(= Bi)$ is small (< 0.1)



In time interval dt the heat transfer dQ from the body is :-

$$dQ = hA\left(T - T_f\right)dt = mC_p dT$$

Putting q = T-T_f, dq = dT, and we may write:

$$\frac{dq}{q} = \frac{hA}{mC_p}dt$$

Integrating gives:- $$\ln\left(\frac{q}{q_0}\right) = \frac{hA}{mC_p}(t - t_0)$$

NB: q$_0$ is the temperature difference at time t_0.

or: $$\boxed{q = q_0 e^{-\frac{t}{t^*}}}$$ [$t_0 = 0$]

ie an exponential change in temperature with time, where the time constant, $t^* = \frac{mC_p}{hA}$

If heat is generated internally:-

$$\boxed{q = q_{\text{inf}}(1 - e^{-\frac{t}{t^*}})}$$ [& $\dot{Q}_{int} = h\,A\,(T_{inf} - T_f)$]