*4th Neural Coding Workshop, Plymouth, UK 10-15 September 2001**

Bifurcations in synaptically coupled Hodgkin-Huxley neurons with a periodic α-function train

Kunichika Tsumoto, Tetsuya Yoshinaga†, Kazuyuki Aihara‡ *and* Hiroshi Kawakami

Faculty of Engineering and †School of Medical Sciences, The University of Tokushima, 2-1 Minami-josanjima, 770-8506; and †3-18-15 Kuramoto 770-8509 Tokushima, JAPAN.
{tsu@ee, †yosinaga@medsci, kawakami@ee}.tokushima-u.ac.jp
‡Faculty of Engineering, The University of Tokyo, 7-3-1 Hongo, Bunkyo-ku, 113-8656 Tokyo, JAPAN; and CREST, Japan Science and Technology Co. (JST), 4-1-8 Motomachi, Kawaguchi 332-0012 Saitama, JAPAN.
‡aihara@sat.t.u-tokyo.ac.jp

1. Introduction

Recently, many arguments for the neural coding in the brain was carried out from various points of view. Especially, in physiological experiments by using a cat[1] or a monkey[2], it was reported that a synchronous firing of neurons can be observed in the visual cotex in the brain. These experimental results are raised the possibility that the various synchronous firing patterns were utilized to do a visual information processing in the brain. For these reasons, it is important to study a mechanism of the generation of synchronization phenomena in neurons.

In this paper, we study various synchronization phenomena observed in two-coupled Hodgkin-Huxley (H-H) neurons[3] with a periodic α-function[4] train. When both the coupling coefficient and the intensity of the periodic α-function train vary, we observe a phenomenon such that a firing rate becomes a monotone increasing function with a staircase structure with respect to a system parameter. Moreover, we obtain parameter regions in which several periodic solutions with mode-locking patterns are observed. Especially, a mechanism of the generation of fundamental firing periodic solutions is clarified by numerical bifurcation analyses.

2. Model Equations

The dynamical system that we consider in this paper is a two-coupled H-H system consisting of the ith H-H equation[3] and a linear differential equation

$$\begin{aligned} C_M \dot{V}^{[i]} &= I_{\mathrm{ion}}^{[i]} + I_{\mathrm{ext}}^{[i]} + I_{\mathrm{syn}}^{[i]} \\ \dot{x}^{[i]} &= \alpha_x(V^{[i]})(1-x^{[i]}) - \beta_x(V^{[i]})x^{[i]} \\ \dot{a}^{[i]} &= b^{[i]}/\tau \\ \dot{b}^{[i]} &= -2b^{[i]}/\tau - a^{[i]}/\tau \end{aligned} \tag{1}$$

for $x = m, h, n$ and $i = 1, 2$, where $\dot{} = d/dt$. Note that the solution of the variable $a^{[i]}$ in Eq.(1) with initial condition $\left(a^{[i]}, b^{[i]}\right) = (0, 1)$ at $t = 0$ represents the α-function or $a^{[i]}(t) = (t/\tau)e^{-t/\tau}$[4], which is a model for describing the time-dependent conductance of the synapse. In Eq.(1), the following definition is used.

$$I_{\mathrm{ion}}^{[i]} = -\overline{g_{\mathrm{Na}}}\{m^{[i]}\}^3 h^{[i]}(V^{[i]} - V_{\mathrm{Na}}) - \overline{g_{\mathrm{K}}}\{n^{[i]}\}^4(V^{[i]} - V_{\mathrm{K}}) - \overline{g_{\mathrm{l}}}(V^{[i]} - V_{\mathrm{l}}) \tag{2}$$

$$I_{\text{ext}}^{[i]} = I_{\text{const}} - H_{\text{syn}}^{[i]}(V^{[i]} - V_{\text{syn}})a_p(t, \tau, \tau_\alpha) \tag{3}$$

$$I_{\text{syn}}^{[i]} = -\sum_{j \neq i} G_{\text{syn}}(V^{[i]} - V_{\text{syn}})a^{[j]} \tag{4}$$

where $a_p(t, \tau, \tau_\alpha)$ represents the periodic α-function train with a period $\tau_\alpha = 2\pi/\omega$.

Each vector $(a^{[i]}, b^{[i]})$ jumps to the constant $(0, 1)$ at $t = t_0^{[i]} + \tau_d$ where $t_0^{[i]}$ is the time when $V^{[i]}$ changes to $V^{[i]} > -30$[mV]. Namely, the firing information of a neuron transforms to another neuron with the time delay τ_d.

3. Results

We consider the two-coupled H-H neurons with a periodic α-function train.

Before showing results, we summarize notations used in bifurcation diagram: G_ℓ^m and I_ℓ^m for tangent bifurcation and period-doubling bifurcation, respectively, where m indicates a bifurcation set for m-periodic point, and ℓ indicates the number to distinguish the several same sets, if they exist.

In the following, several system parameters except for the coupling coefficient G_{syn} and the intensity of the input, $H_{\text{syn}}^{[2]}$, in Eqs.(1)–(4) are fixed as

$$C_M = 1, \quad V_{\text{K}} = -77, \quad V_{\text{Na}} = 50, \quad V_{\text{l}} = -54.4, \quad \overline{g_{\text{K}}} = 36, \quad \overline{g_{\text{Na}}} = 120,$$
$$\overline{g_{\text{l}}} = 0.3, \quad I_{\text{const}} = 5, \quad \tau = 0.2, \quad \tau_d = 0.32, \quad \omega = 0.505, \quad V_{\text{syn}} = 0, \quad H_{\text{syn}}^{[1]} = 1.$$

By varying values of the system parameters $H_{\text{syn}}^{[2]}$ and G_{syn}, we can observe periodic solutions with various types of mode-locking patterns, see Figs.1(a)–(e) for examples of typical stable firing periodic solutions. To discuss properties of these firing periodic solutions with several mode-locking patterns, we define the mean firing rate[5] of the neuron $N^{[i]}$ for $i = 1, 2$, as the ratio of output to input pulse. Figure 2(a) shows a 2-parameter diagram of firing rates for the neuron $N^{[2]}$ by varying $H_{\text{syn}}^{[2]}$ and G_{syn}. In addition, a graph of the firing rate of the neuron $N^{[i]}$ for $i = 1, 2$ is shown in Fig.2(b). We see that it shows a monotone increasing function of the system parameter with a staircase structure.

Next, we consider bifurcations for fundamental periodic firing solutions. Figure 3(a) shows a bifurcation diagram of 1-periodic solutions in $(H_{\text{syn}}^{[2]}, G_{\text{syn}})$ plain. Examples of different types of 1-periodic solutions are shown in Fig.3(b). In Fig.3(a), the regions shaded by the patterns , , and denote parameters at which stable firing 1-periodic solutions with $0:1$ mode-locking pattern, $1:1$ mode-locking pattern, shifted $1:1$ mode-locking pattern (I) and shifted $1:1$ mode-locking pattern (II) exist, respectively. The regions overlapped by several patterns denote coexistence of the corresponding solutions, depending on the initial condition.

We consider bifurcations of the one side firing periodic solution with the $0:1$ mode-locking pattern. For the periodic solution, the neuron $N^{[1]}$ fires and another does not. By increasing the value of G_{syn} through the period-doubling bifurcation set I_1^1, for fixed value of $H_{\text{syn}}^{[2]}$ as, e.g., 0.1, a stable 2-periodic solution generates. The bifurcated stable 2-periodic solution is also one side firing periodic solution, or the neuron $N^{[2]}$ is non-firing. It is conjectured that the transition between one side firing and both sides firing is caused by an abrupt change.

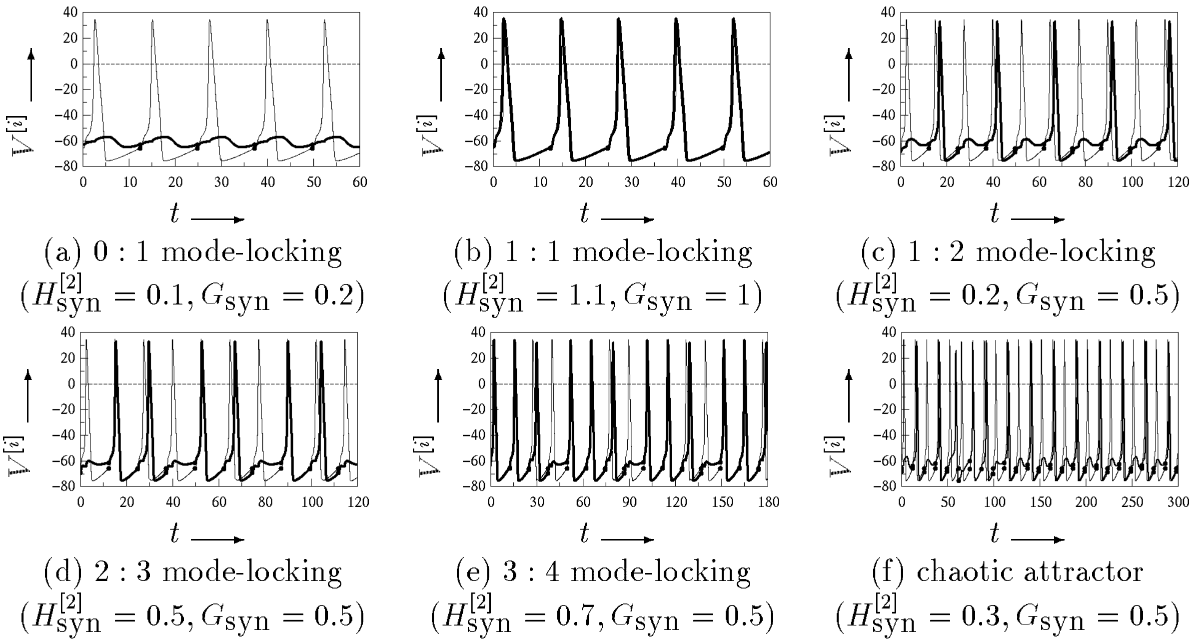


Figure 1: Oscillations observed in Eq.(1). The light and heavy lines indicate the variables $V^{[1]}$ and $V^{[2]}$, respectively. The circled points denote iterated points by the Poincaré map.

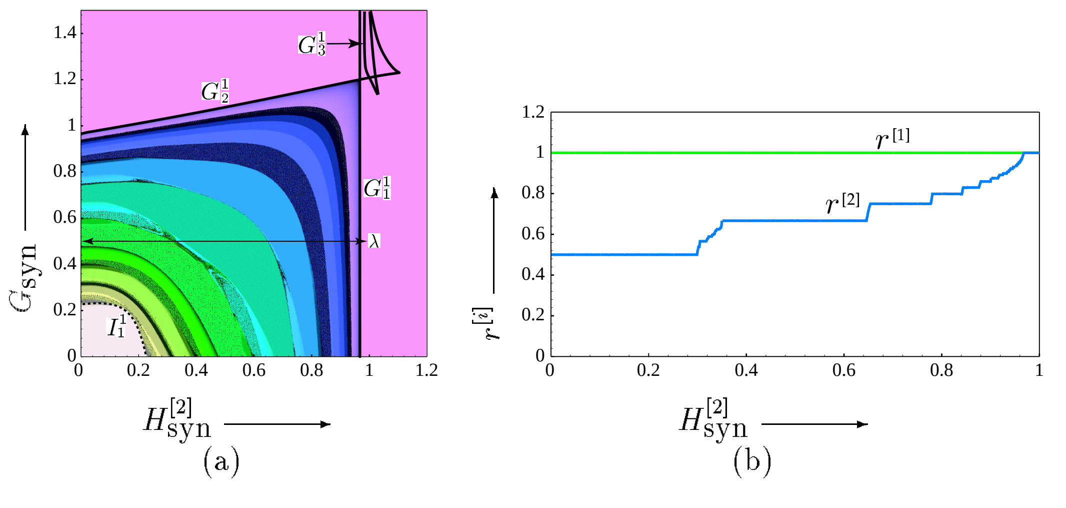
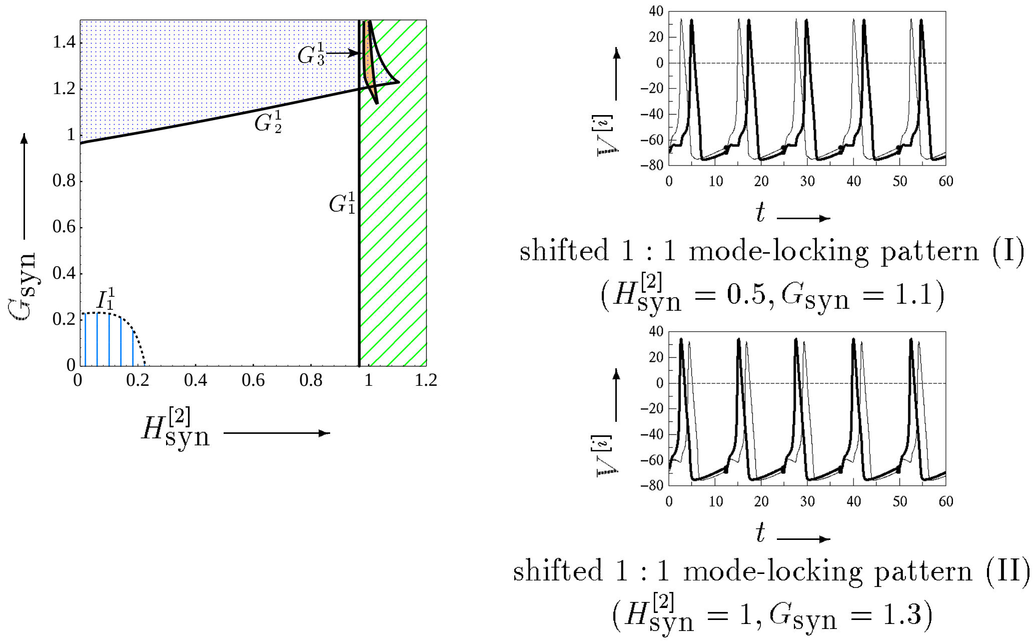


Figure 2: (a) A two-parameter diagram for firing rates of the neuron $N^{[2]}$ and (b) a graph of firing rate along the line λ in the diagram (a), at $G_{\rm syn} = 0.5$.



(a) (b)

Figure 3: (a) Bifurcation diagram for periodic solutions and (b) examples of two types of shifted 1 : 1 mode-locked solutions with different leading phase patterns.

4. Conclusion

We have investigated mechanisms of various bifurcation phenomena observed in synaptically two-coupled H-H neurons with a periodic α-function train. We calculated bifurcations of fundamental synchronous firing solutions with various mode-locking patterns. We found mechanisms of transitions among various types of firing periodic solutions.

To investigate bifurcation structure for periodic solutions with higher order of period is an interesting future problem.

References

[1] C.M. Gray, P. Konig, A.K. Engel and W. Singer, “ Oscillatory responses in the cat visual cotex exhibit inter-columnar synchronization which reflects global stimulus properties”, Nature, Nature, vol.338, No. 23, pp. 334–337, Mar 1989.

[2] E. Vaadla, I. Haalman, M. Abeles, H. Bergman, Y. Prut, H. Slovin and A. Aertsen, in monkey cortex in relation to behavioural events”, Nature, vol. 373, No. 9, pp. 515–518, Feb 1995.

[3] A.L. Hodgkin and A.F. Huxley, “A qualitative description of membrane current and its application to conduction and excitation in nerve,” *J. Physiol.*, vol.117, pp.500–544, 1952.

[4] E.R. Kandel, J.H. Schwartz and T.M. Jessel, “Synaptic transmission,” in *Principles of Neural Science*, 3rd edition, eds. E.R. Kandel, J.H. Schwartz and T.M. Jessel, Appleton and Lange, Norwalk, Chap.9.

[5] S. Doi and S. Sato, “The global bifurcation structure of the BVP neuronal model driven by periodic pulse trains,” *Mathematical Biosciences*, Vol. 125, pp.229–250, 1995.